D-branes on general $\mathcal{N} = 1$ backgrounds: superpotentials and D-terms

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ABSTRACT: We study the dynamics governing space-time filling D-branes on Type II flux backgrounds preserving four-dimensional $\mathcal{N} = 1$ supersymmetry. The four-dimensional superpotentials and D-terms are derived. The analysis is kept on completely general grounds thanks to the use of recently proposed generalized calibrations, which also allow one to show the direct link of the superpotentials and D-terms with BPS domain walls and cosmic strings respectively. In particular, our D-brane setting reproduces the tension of D-term strings found from purely four-dimensional analysis. The holomorphicity of the superpotentials is also studied and a moment map associated to the D-terms is proposed. Among different examples, we discuss an application to the study of D7-branes on SU(3)-structure backgrounds, which reproduces and generalizes some previous results.

KEYWORDS: Flux compactifications, Supersymmetric Effective Theories, D-branes.

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Contents

1. Introduction

The study of string theory compactifications to four dimensions with non-trivial fluxes is not only interesting by itself, but seems to be necessary if we hope to use string theory to describe realistic scenarios. Moreover, backgrounds with fluxes arise naturally also in the context of the gauge-gravity correspondence. D-branes play a central role in several aspects of these constructions and thus the study of their properties on nontrivial flux backgrounds is of both formal and phenomenological interest.

In this paper we study the dynamics of space-time filling D-branes in the most general Type II backgrounds preserving four-dimensional Poincaré invariance and $\mathcal{N} = 1$ super-symmetry. The aim is that of presenting a unified analysis that automatically includes a

large class of cases, having $\mathcal{N} = 1$ background supersymmetry as the unique requirement. This analysis obviously includes as special subcases the $\mathcal{N} = 2$ backgrounds that are obtained by turning off the Ramond-Ramond (RR) fields, and in particular the limit in which the internal space reduces to a standard Calabi-Yau space.

D-brane dynamics in $\mathcal{N} = 2$ compactifications on standard Calabi-Yau three-folds constitute an active past and present topic of research (for reviews and complete lists of references see for example [1, 2]). One approach, that we will follow in the present paper, is to consider D-branes filling the four flat directions and wrapping some internal cycle, describing the system by an effective four-dimensional $\mathcal{N} = 1$ theory. The well-known geometrical properties of the underlying Calabi-Yau spaces allow one to employ a series of familiar technics. Many problems can be addressed systematically by using the two integrable structures of the Calabi-Yau, i.e. the complex and symplectic structures, and for example the associated twisted topological theory constitutes an efficient way to inspect the holomorphic sector of the theory [3].

In general, the reduction of the background supersymmetry to $\mathcal{N} = 1$, obtained by giving non-trivial expectation value to the internal fluxes, can drastically change the geometry of the internal space.[1] In particular, the symplectic and complex structures cease to be defined in general and, even in cases when they are both defined, they may not be simultaneously integrable. However, as discussed in [5] for a wide class of $\mathcal{N} = 1$ vacua, the minimal supersymmetry still imposes an integrable structure on the internal manifold that can be described as a generalized complex structure by using the language of generalized complex geometry [6, 7]. The complex and symplectic structures are substituted by a pair of pure spinors (that are particular kinds of polyforms - formal sums of forms of different degrees) of definite and opposite parity, that are associated to corresponding generalized *almost* complex structures. The background supersymmetry conditions are written in terms of these two pure spinors and imply that one of the associated generalized almost complex structures is actually integrable, while the RR background fluxes provide an obstruction to the integrability of the other.

In this paper we will consider the most general class of $\mathcal{N} = 1$ backgrounds admitting supersymmetric static D-brane configurations. These backgrounds constitute a subclass of the vacua analyzed in [5] and we will refer to them as *D-calibrated* backgrounds. The name is justified by the fact that, as shown in [8], these supersymmetric backgrounds can be completely characterized in terms of a new kind of generalized calibrations associated to the possible supersymmetric static D-brane configurations (i.e. filling two, three or all four space-time directions).[2] The generalized calibrations are essentially given by the real and imaginary parts of the background pure spinors, and provide an elegant physical interpretation for them.

Introducing a space-time filling D-brane wrapping some internal generalized cycle (defined as cycle with a world-volume field-strength on it) on these D-calibrated backgrounds, the effective four-dimensional description should admit an $\mathcal{N} = 1$ structure. Indeed, we

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[1]For a review on flux compactifications see for example [4].

[2]Similar generalized calibrations were introduced in [9] for the subclass of $\mathcal{N} = 2$ backgrounds obtained by switching off the RR fields.

will show how it is possible to identify superpotentials and D-terms that can be written in a completely general form in terms of the underlying generalized calibrations (and then of the background pure spinors). The associated F-flatness and D-flatness conditions are equivalent to the supersymmetry/calibration condition found in [8].

Regarding the superpotential, we will see how it only involves the background *integrable* pure spinor and the associated F-flatness condition requires that the D-brane must wrap a *generalized complex submanifold*, as defined in [7]. This result can be seen as an extension of the "decoupling statement" of [10], that in the present context can be rephrased as the requirement that the superpotential only 'sees' the underlying (integrable) generalized complex structure. The superpotentials we find may be adopted for the topological branes [11 – 15] of the associated topological models [5, 11, 16]. Our superpotentials generalize known superpotentials for D-branes on Calabi-Yau manifolds, as studied for example in [3, 17 – 19]. They are also in agreement with previous results for D7-branes with worldvolume and/or background fluxes [20, 21]. We will discuss the holomorphic properties of the superpotentials and shall see how they can be addressed in a unified way, again generalizing previous results for D-branes on Calabi-Yau spaces (see for example the discussion of [22, 23]).

It is well known that the tension of a possible BPS domain wall in an $\mathcal{N} = 1$ theory is expressed uniquely in term of the superpotential. This relation has been used for example in [24, 25] for deriving flux induced superpotentials for the closed string moduli. Using the underlying generalized calibrations, we will see how the same approach can also be used to give an alternative and more physical derivation of the D-brane superpotentials, thus obtaining a non-trivial consistency check of our results.

Regarding the D-flatness condition, in the standard Calabi-Yau case, it can be seen as a deformed Hermitian-Yang-Mills equation for the holomorphic connection on the holomorphic B-cycles, while for Lagrangian A-branes it corresponds to the additional "speciality" conditions (a discussion and more references can be found in [1]). These conditions are equivalent to the vanishing of a moment map associated to the U(1) gauge symmetry on the D-brane through an appropriate symplectic structure on the configuration space [22, 23]. The vanishing of the moment map provides a transversal slicing for the imaginary extension of the gauge group action, whose complexification is a symmetry of the superpotential. An extension of this approach to the case of SU(3)-structure backgrounds has been discussed in [26]. We will propose a symplectic form that generalizes the known symplectic structures of the above mentioned particular subcases to our more general setting. Using this, the U(1) gauge symmetry on the wrapped cycle is associated to a moment map whose vanishing condition is equivalent the our general D-flatness condition.

The insight given by the generalized calibrations characterizing these backgrounds allows us to derive another interesting physical result. Namely, the D-term turns out to be strictly related to the BPS cosmic strings obtained by wrapping D-branes filling only two flat space-time directions around internal generalized cycles. First, we will discuss how the D-flatness condition can be satisfied only if a certain topological constant ξ vanishes. This constant can be identified with the Fayet-Iliopoulos term in the effective four-dimensional description. Then, we will see how the BPS cosmic string tension computed from our D-

brane setting matches precisely the BPS cosmic string tension $T_{\text{string}} = 2\pi\xi$ obtained from $\mathcal{N} = 1$ supergravity in [27], which should describe a D$\bar{\text{D}}$-brane pair. This result provides a non-trivial check of the identification proposed in [27] between these D-term supergravity string solitons and the effective cosmic strings obtained by wrapping D-branes.

The plan of the paper is as follows. In section 2 we review the basic results of [8], i.e. the D-brane supersymmetry conditions and the associated generalized calibrations. In section 3 we show how, starting from the Dirac-Born-Infeld (DBI) plus Chern-Simons (CS) action for D-branes, we can organize the four-dimensional potential in an explicit $\mathcal{N} = 1$ form, recognizing the supersymmetry/calibration conditions in the F- and D-flatness conditions of the four-dimensional description. In section 4 we introduce a superpotential that gives rise to the F-flatness condition. This can be written in a universal way by using the underlying integrable pure spinor. In section 5 we give an alternative derivation of the superpotential by using domain wall D-brane configurations. Cosmic string D-brane configurations are considered in section 6, stressing their relation with the Fayet-Iliopoulos contribution to the D-term and giving a general nontrivial argument in favor of their identification with the supergravity cosmic strings constructed in [27]. The holomorphicity properties of the superpotentials are studied in section 7, where an almost complex structure is introduced on the D-brane configuration space from the SU(3) × SU(3) structure of the underlying background. In section 8 we discuss the reduction of this almost complex structure to the superpotential critical subspace. In section 9 we turn to the D-flatness condition and see how it can be interpreted as the vanishing of a moment map associated to the world-volume gauge symmetry by an appropriately defined symplectic structure. Section 10 is dedicated to some explicit examples in the more specific case of backgrounds with internal SU(3)-structure [28]: we will consider D3, D5, D6 and D7-branes, with particular attention being paid to the last case, for which our general analysis reproduces results present in the literature (see for example [20, 21, 29 – 32]). Finally, in section 11 we present our concluding remarks. Appendix A contains a more detailed discussion on our parametrization of the infinitesimal deformations of generalized cycles.

2. D-calibrated $\mathcal{N} = 1$ vacua

In this paper we consider the most general Type II $\mathcal{N} = 1$ backgrounds with four-dimensional Poincaré invariance which admit possible supersymmetric D-branes filling one or more flat space directions and wrapping some internal cycle. As discussed in [8], all the backgrounds satisfying these conditions consists of a subclass of the family of $\mathcal{N} = 1$ vacua analyzed in [5], that we refer to as *D-calibrated* since they can be completely characterized by the existence a kind of generalized calibrations [8] as we are going to review in this section.

Let us discuss briefly the main properties of the $\mathcal{N} = 1$ D-calibrated backgrounds, following the conventions of [8]. The ten dimensional metric can be written in the general form

$$ds^2 = e^{2A(y)}dx^\mu dx_\mu + g_{mn}(y)dy^m dy^n \,, \tag{2.1}$$

where x^μ, $\mu = 0, \ldots, 3$ label the four-dimensional flat space, and y^m, $m = 1, \ldots, 6$, the internal space. The B-form field-strength $H = dB$ can have legs only along internal directions, while the generalized RR field-strengths

$$F_{(n+1)} = dC_{(n)} + H \wedge C_{(n-2)} \, , \tag{2.2}$$

are allowed to have the restricted form

$$F_{(n)} = \hat{F}_{(n)} + Vol_{(4)} \wedge \tilde{F}_{(n-4)} \, . \tag{2.3}$$

All the fields appearing in this ansatz (including the dilaton Φ) can depend only on the internal coordinates y^m. Note also that the usual electric-magnetic Hodge duality relating lower and higher degree RR field-strength translates into the relation $\tilde{F}_{(n)} = (-)^{\frac{(n-1)(n-2)}{2}} *_6 \hat{F}_{(6-n)}$ between their internal components.

Starting from this bosonic ansatz, the $\mathcal{N} = 1$ supersymmetry imposes that there exist four independent ten dimensional Killing spinors that can be written in terms of an arbitrary four-dimensional constant spinor ζ_+ of positive chirality and two internal six-dimensional spinors $\eta^{(1)}$ and $\eta^{(2)}$. The resulting Killing equations give strong constraints on the background bosonic ansatz. The important result proved in [5] is that these supersymmetry constraints on the background fields can be nicely written in terms of the following two polyforms of definite parity

$$\Psi^+ = \sum_{k \geq 0} \Psi^+_{(2k)} \qquad \Psi^- = \sum_{k \geq 0} \Psi^-_{(2k+1)}, \tag{2.4}$$

corresponding via the usual Clifford map to the bispinors[3]

$$\slashed{\Psi}^+ = \eta^{(1)}_+ \otimes \eta^{(2)\dagger}_+ \, , \qquad \slashed{\Psi}^- = \eta^{(1)}_+ \otimes \eta^{(2)\dagger}_- \, . \tag{2.5}$$

$\Psi^\pm$ can be seen to be pure spinors in the context of the generalized complex geometry and define corresponding generalized almost complex structures.[4] As we will presently recall, in the $\mathcal{N} = 1$ vacua we are interested in, only one of these two generalized almost complex structure will actually be integrable [5].

Note that not all the possible $\mathcal{N} = 1$ solutions with 4d Poincaré invariance can be studied in these terms. Indeed there are some pure NS $\mathcal{N} = 1$ vacua [34 – 36] that cannot be incorporated in these class of backgrounds, since in these cases one of the two internal spinors vanishes and then both $\Psi^\pm$ vanish as well, spoiling of any mean the above approach. However, as discussed in [8], the only $\mathcal{N} = 1$ backgrounds that can admit supersymmetric (static) D-branes filling one or more flat space directions are those whose internal spinors have the same norm

$$||\eta^{(1)}||^2 = ||\eta^{(2)}||^2 = |a|^2 \, . \tag{2.6}$$

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[3]See [33] for a previous analysis using these bispinors in pure NS backgrounds.

[4]More explicit expressions for the pure spinors $\Psi^\pm$ can be found in [5]. The case of D-calibrated SU(3)-structure vacua, which include the standard $\mathcal{N} = 2$ compactifications on flux-less Calabi-Yau's as a subcase, will be discussed in detail in section 10.

This means that the cases we are interested in can be completely covered by the description given in [5] and the condition (2.6) allows also to characterize this class of $\mathcal{N} = 1$ backgrounds as D-calibrated. More explicitly, taking into account the additional requirement given in (2.6), the supersymmetry conditions for the backgrounds can be split in two parts. One relates the warp factor A to the norm of the internal spinor

$$d|a|^2 = |a|^2 dA \Rightarrow |a|^2 = ce^A,\tag{2.7}$$

for some constant c. This relation is a direct consequence of the 4d $\mathcal{N} = 1$ supersymmetry as it is equivalent to require that $\bar{\epsilon}\Gamma^\mu\epsilon$ (here ϵ is the 10d Killing spinor doublet) must be a (constant) Killing vector generating the 4d spacetime translations.

The other supersymmetry conditions involve the two pure spinors $\Psi^\pm$ characterizing our backgrounds[5]

$$e^{-2A+\Phi}d_H\big(e^{2A-\Phi}\Psi_1\big) = dA \wedge \bar{\Psi}_1 + \frac{i|a|^2}{8}e^\Phi\tilde{F},$$
$$d_H\big(e^{2A-\Phi}\Psi_2\big) = 0,\tag{2.8}$$

where

$$d_H = d + H\wedge\tag{2.9}$$

is the H-twisted differential (such that $d_H^2 = 0$) and for Type IIA we have

$$\Psi_1 = \Psi^-, \qquad \Psi_2 = \Psi^+ \quad \text{and} \quad \tilde{F} = \tilde{F}_A = \tilde{F}_{(0)} + \tilde{F}_{(2)} + \tilde{F}_{(4)} + \tilde{F}_{(6)},\tag{2.10}$$

while for Type IIB

$$\Psi_1 = \Psi^+ \quad, \quad \Psi_2 = \Psi^- \quad \text{and} \quad \tilde{F} = \tilde{F}_B = \tilde{F}_{(1)} + \tilde{F}_{(3)} + \tilde{F}_{(5)}.\tag{2.11}$$

As proved in [8], the equations given in (2.8) can be completely characterized in terms of properly defined generalized calibrations associated to possible static supersymmetric D-branes. These are given by D-branes filling two (strings), three (domain walls) or all four flat space-time directions, and wrapping some internal generalized cycle $(\Sigma, \mathcal{F})$, i.e. a cycle Σ with a possible general world-volume field strength $\mathcal{F}$ on it (which satisfies the Bianchi identity $d\mathcal{F} = P_\Sigma[H]$). The associated generalized calibrations are given by d_H-closed formal sums of (real) forms $\omega = \sum_k \omega_{(k)}$ of definite degree parity which are written in terms of the pure spinors $\Psi^\pm$ and are properly energy minimizing when combined with the world-volume field strength $\mathcal{F}$. More explicitly, for any generalized cycle $(\Sigma, \mathcal{F})$,

$$P_\Sigma[\omega] \wedge e^{\mathcal{F}}|_{\text{top}} \leq \mathcal{E}(\Sigma, \mathcal{F}),\tag{2.12}$$

where $\mathcal{E}(\Sigma, \mathcal{F})$ refers to the energy density of the D-brane wrapping $(\Sigma, \mathcal{F})$. Note that, if on one hand the form of the generalized calibration is completely general for wrapped

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[5]See [8] for the conventions we are using and how they are related to the ones adopted in [5].

cycles of any dimension, on the other hand it *does* depend explicitly on the number of flat space-like directions filled by the D-brane.[6] If we introduce the normalized pure spinors

$$\hat{\Psi}^{\pm} = -\frac{8i}{|a|^2}\Psi^{\pm}\,,\tag{2.13}$$

the generalized calibrations for space-time filling D-branes are given by [8]

$$\omega^{(4d)} = e^{4A}\Big[e^{-\Phi}\mathrm{Re}\hat{\Psi}_1 - \tilde{C}\Big]\,,\tag{2.14}$$

where $\tilde{C} = \sum_k \tilde{C}_{(k)}$ with k even in Type IIB and odd in Type IIA, and $\tilde{C}_{(k)}$ are potentials for $\tilde{F}_{(k+1)}$, such that $\tilde{F}_{(k+1)} = d_H\tilde{C}_{(k)} + 4dA \wedge \tilde{C}_{(k)}$. The generalized calibrations for four-dimensional strings and domain walls are given by

$$\omega^{(\mathrm{string})} = e^{2A-\Phi}\mathrm{Im}\hat{\Psi}_1\,,\qquad \omega^{(\mathrm{DW})} = e^{3A-\Phi}\mathrm{Re}(e^{i\theta}\hat{\Psi}_2)\,.\tag{2.15}$$

The calibration for the domain walls contain an a priori arbitrary phase specifying the preserved half of the four-dimensional $\mathcal{N} = 1$ supersymmetry. The inequality (2.12) for each of the generalized calibrations comes from completely algebraic considerations while the differential requirement that they are d_H-closed is completely equivalent to the background supersymmetry conditions given in (2.8). Note also that the generalized calibrations for the space-time filling and four-dimensional string D-branes involve the *non-integrable* pure spinor, while the generalized calibration for four-dimensional domain wall D-branes involves the *integrable* pure spinor.

A supersymmetric D-brane configuration can be completely characterized as a D-brane wrapping a *generalized calibrated cycle*, i.e. a generalized cycle $(\Sigma, \mathcal{F})$ which saturates in each point the upper bound in (2.12). As discussed in [8], this condition can be split in an equivalent pair of conditions. In the case of the space-time filling D-branes (on which we focus from now on), these are given by

$$\begin{aligned}
P[dy^m \wedge \Psi_2 + g^{mn}\imath_n\Psi_2] \wedge e^{\mathcal{F}}|_{\mathrm{top}} &= 0\,, &\qquad \mathrm{F-flatness}\,,\\
P[\mathrm{Im}\Psi_1] \wedge e^{\mathcal{F}}|_{\mathrm{top}} &= 0\,, &\qquad \mathrm{D-flatness}\,.
\end{aligned}\tag{2.16}$$

The reason why we have used the names F-flatness and D-flatness will be the focus of the following discussions. For the moment, let us only recall that the F-flatness imposes that the D-brane must wrap a generalized complex submanifold [8] and then specifies the generalized complex geometry of the supersymmetric D-branes. In the SU(3)-structure cases, where the internal manifold is either complex (IIB) or symplectic (IIA), this requirement is completely equivalent [7] to require that the D-brane must be holomorphically embedded with $\mathcal{F}_{(2,0)} = 0$ in Type IIB, and must wrap Lagrangian or more general coisotropic [37] generalized cycles in Type IIA. This is completely analogous to what happens in the flux-less Calabi-Yau case, where the above geometrical conditions must be supplemented by a stability condition

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[6]This effect comes directly from the $\mathcal{N} = 1$ supersymmetry of the background (that, for example, implies a nontrivial warp factor). In the $\mathcal{N} = 2$ limit reached by turning off the RR fields, the form of the generalized calibrations acquires an arbitrary phase [9] and does not depend on the filled flat directions.

which can be read as a deformed Hermitian-Yang-Mills equation for B branes and the "speciality" condition for Lagrangian A-branes and their coisotropic generalization [38, 39]. As discussed in a series of paper by Douglas and collaborators (see e.g. [1] for a review), this stability condition can be seen as a D-flatness condition, obtained by imposing the vanishing of an associated moment map. In the following sections we will discuss in detail the above F-flatness and D-flatness in our general setting considering $\mathcal{N} = 1$ backgrounds, trying to clarify their meaning and their relation with the results already known in the Calabi-Yau case.

3. The four-dimensional point of view

In this section, using the results of [8] reviewed in section 2, we would like to pass to a four-dimensional description of the dynamics of the space-time filling D-branes, which should be ultimately described by a four-dimensional $\mathcal{N} = 1$ effective theory.

Let us start by deriving a form which depends explicitly on the pure spinors $\Psi^{\pm}$ for the potential $\mathcal{V}(\Sigma, \mathcal{F})$ associated to a space-time filling D-brane wrapping the generalized cycle $(\Sigma, \mathcal{F})$. Consider a D-brane wrapping an n-dimensional cycle Σ and introduce a complex F-term vector density $\mathcal{W}_m$, a real D-term density $\mathcal{D}$ and the scalar density Θ in the following way

$$\mathcal{W}_m d\sigma^1 \wedge \cdots \wedge d\sigma^n = \frac{(-)^{n+1}}{2} P[e^{3A-\Phi}(\imath_m + g_{mk}dy^k\wedge)\hat{\Psi}_2] \wedge e^{\mathcal{F}}|_{\text{top}} ,$$
$$\mathcal{D}d\sigma^1 \wedge \cdots \wedge d\sigma^n = P[e^{2A-\Phi}\text{Im}\hat{\Psi}_1] \wedge e^{\mathcal{F}}|_{\text{top}} ,$$
$$\Theta d\sigma^1 \wedge \cdots \wedge d\sigma^n = P[e^{4A-\Phi}\text{Re}\hat{\Psi}_1] \wedge e^{\mathcal{F}}|_{\text{top}} . \tag{3.1}$$

Note that, if we are not "too far" from a supersymmetric configuration (which has also an appropriate orientation on Σ), we can assume that $\Theta > 0$. From the discussion presented in [8], we can argue that

$$\sqrt{\det(P[g] + \mathcal{F})} = e^{-4A+\Phi}\sqrt{\Theta^2 + e^{4A}\mathcal{D}^2 + 2e^{2A}g^{mn}\mathcal{W}_m\bar{\mathcal{W}}_n} . \tag{3.2}$$

The complete four-dimensional potential for the D-brane is then given by

$$\mathcal{V}(\Sigma, \mathcal{F}) = \int_{\Sigma} d^n\sigma \sqrt{\Theta^2 + e^{4A}\mathcal{D}^2 + 2e^{2A}g^{mn}\mathcal{W}_m\bar{\mathcal{W}}_n} - \int e^{4A}\tilde{C} \wedge e^{\mathcal{F}} . \tag{3.3}$$

This potential contains the full nonlinear (static) interactions governing the D-brane. We want now to consider the expansion of such a potential around a supersymmetric vacuum configuration $(\Sigma_0, \mathcal{F}_0)$, which is characterized by the condition $\mathcal{W}_m(\Sigma_0, \mathcal{F}_0) = \mathcal{D}(\Sigma_0, \mathcal{F}_0) = 0$. Then, we can consider very small $\mathcal{W}_m$ and $\mathcal{D}$ and expand the square root in the potential (3.3). As a result, at the quadratic order in $\mathcal{W}^m$ and $\mathcal{D}$ we obtain the following potential

$$\mathcal{V}(\Sigma, \mathcal{F}) \simeq \int_{\Sigma} P[\omega^{(4d)}] \wedge e^{\mathcal{F}} + \frac{1}{2}\int_{\Sigma} \frac{d^n\sigma}{\Theta}(e^{4A}\mathcal{D}^2 + 2e^{2A}g^{mk}\mathcal{W}_m\bar{\mathcal{W}}_k)$$
$$= \mathcal{V}(\Sigma_0, \mathcal{F}_0) + \frac{1}{2}\int_{\Sigma} \frac{d^n\sigma}{\Theta}(e^{4A}\mathcal{D}^2 + 2e^{2A}g^{mk}\mathcal{W}_m\bar{\mathcal{W}}_k) , \tag{3.4}$$

where in the last step we have used the d_H-closedness of the generalized calibration $\omega^{(4d)}$ defined in (2.14).

In order to better identify this potential with the standard potential of $\mathcal{N} = 1$ gauge theories, we must introduce metrics on the spaces[7] $\Gamma(\Sigma, \mathbb{R})$ (that can be identified with the Lie algebra of the four-dimensional gauge group $U(1)^\infty$) and $\Gamma(T_M|_\Sigma)$. Let us consider first two world-volume functions $f, g \in \Gamma(\Sigma, \mathbb{R})$. We define

$$k(f,g) \equiv \int_\Sigma fg P[e^{-\Phi} \mathrm{Re}\hat{\Psi}_1] \wedge e^{\mathcal{F}} . \tag{3.5}$$

This can be easily seen to be the natural metric for the Lie algebra of the gauge group by expanding the DBI action to write the kinetic term for the four-dimensional field-strength. Secondly, we introduce the following metric on $\Gamma(T_M|_\Sigma)$

$$G(X,Y) \equiv \int_\Sigma g_{mn} X^m Y^n P[e^{2A-\Phi} \mathrm{Re}\hat{\Psi}_1] \wedge e^{\mathcal{F}} , \tag{3.6}$$

that on the other hand defines the natural metric on the space of four-dimensional scalars (this will become more evident from the following discussions). The above metrics are non-degenerate for generalized cycles $(\Sigma, \mathcal{F})$ not "too far" from the supersymmetric ones, for which

$$P[\mathrm{Re}\hat{\Psi}_1] \wedge e^{\mathcal{F}}|_{\mathrm{top}} = \frac{\sqrt{\det(P[g] + \mathcal{F})}}{\sqrt{\det P[g]}} d\mathrm{Vol}_\Sigma . \tag{3.7}$$

We can now consider the densities $\mathcal{D}$ and $\mathcal{W}_m$ as belonging to the dual of $\Gamma(\Sigma, \mathbb{R})$ and $\Gamma(T_M|_\Sigma)$ by using the natural pairing given by the ordinary integration (if for example $f \in \Gamma(\Sigma, \mathbb{R})$ and θ is a dual density, $\langle \theta, f \rangle = \int_\Sigma f\theta$). Thus, we can write the potential (3.4) in the form

$$\mathcal{V}(\Sigma, \mathcal{F}) \simeq \mathcal{V}(\Sigma_0, \mathcal{F}_0) + \frac{1}{2} k^{-1}(\mathcal{D}, \mathcal{D}) + G^{-1}(d\mathcal{W}, d\bar{\mathcal{W}}) , \tag{3.8}$$

where the $\mathcal{W}_m$'s have been considered as the components of the formal object $d\mathcal{W} = \mathcal{W}_m dx^m$. As it will be clear from the following sections, we can really think to $d\mathcal{W}$ as a differential of a proper superpotential $\mathcal{W}$. Thus, in the expansion of $\mathcal{V}$ given in (3.8) one can recognize a contribution $\mathcal{V}(\Sigma_0, \mathcal{F}_0)$, which can be seen as a zero point energy, plus a term that is formally identical to the standard potential of the $\mathcal{N} = 1$ theories, given by the sum of the squares of the D- and F-terms.

In order to identify more explicitly $\mathcal{W}_m$ and $\mathcal{D}$ as the actual F- and D-terms of the four-dimensional $\mathcal{N} = 1$ description for space-time filling D-branes, we can give a look at the form of the supersymmetry transformations of the world-volume fermions. This can be obtained by gauge fixing the κ-symmetry of the D-brane superactions [40, 41]. Here we will use the conventions of [42], consistently with [8]. In particular, we will use the covariant κ-fixing explicitly discussed in [42], where the second Majorana-Weyl spinor is

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[7]We indicate with $\Gamma(E)$ the space of sections of a vector bundle E and with $\Gamma(N, \mathbb{R})$ the space of real functions on a manifold N.

put to zero. Then, the D-brane fermionic degrees of freedom are described by a single ten dimensional Majorana-Weyl spinor θ and a background Killing spinors ε induces a corresponding supersymmetry transformation for θ. When we specialize to the supersymmetry transformations of an $\mathcal{N} = 1$ vacuum configuration, i.e. with constant fields preserving Poincaré symmetry, these are given by

$$\delta_\varepsilon \theta = \zeta_+ \otimes \left[\left(1 - \frac{e^{\Phi-4A}\Theta}{\sqrt{\det(P[g] + \mathcal{F})}} + i\frac{e^{\Phi-2A}\mathcal{D}}{\sqrt{\det(P[g] + \mathcal{F})}} \right) \eta_+^{(1)} + \right.$$
$$\left. + \frac{e^{\Phi-3A}g^{mn}\mathcal{W}_n}{\sqrt{\det(P[g] + \mathcal{F})}} \hat{\gamma}_m \eta_-^{(1)} \right] + \text{ c.c. }, \tag{3.9}$$

where ζ_+ is an arbitrary constant four-dimensional spinor of positive chirality generating the $\mathcal{N} = 1$ supersymmetry (see [8] for more details on the notation). If we now consider $\mathcal{W}_m$ and $\mathcal{D}$ very small (around a supersymmetric configuration), the above supersymmetry transformation becomes at leading order

$$\delta_\varepsilon \theta \simeq i e^{-2A}\mathcal{D}^*(\zeta_+ \otimes \eta_+^{(1)}) + e^{-A}\mathcal{W}^{*n}(\zeta_+ \otimes \hat{\gamma}_m \eta_-^{(1)}) + \text{ c.c. }, \tag{3.10}$$

where $\mathcal{D}^* \in \Gamma(\Sigma, \mathbb{R})$ and $\mathcal{W}^* \in \Gamma(T_M|_\Sigma)$ are the duals associated to $\mathcal{D}$ and $d\mathcal{W}$ by using the metrics (3.5) and (3.6) respectively. In order to give a four-dimensional interpretation of the above supersymmetry transformation, let us first have a look at the bosonic field content living on the D-brane.

By starting from a fixed world-volume field strength configuration $\mathcal{F}$ on the internal cycle Σ, the world-volume gauge field fluctuations split in two parts: $a_\mu(x, \sigma)$, with indices along the four flat directions, and $a_\alpha(x, \sigma)$ with indices along the internal directions wrapped by the cycle. On the other hand, the fluctuations of the brane can be described by sections $\phi^{\hat{m}}(x, \sigma) \in \Gamma(T_\Sigma^\perp)$ of the orthogonal bundle. Note that all the fields depend on σ, since we are not doing a real dimensional reduction. So, we can think of them as containing an infinite set of four-dimensional fields. In particular, $a_\mu(x, \sigma)$ gives rise to an infinite set of four-dimensional $\mathcal{N} = 1$ abelian vector multiplets, once completed with the corresponding gaugini that we indicate schematically with $\lambda(x, \sigma)$. On the other hand $a_\alpha(x, \sigma)$ and $\phi^{\hat{m}}(x, \sigma)$ should combine to form the bosonic content of an infinite number of chiral multiplets whose fermionic components we indicate with χ^m.

We can now consider more closely the $\text{SU}(3) \times \text{SU}(3)$ structure associated to the internal manifold M. Each $\text{SU}(3)$ factor gives a different reduction of the $\text{SO}(6)$ structure group of T_M, and is an independent symmetry acting separately on $\eta^{(1)}$ and $\eta^{(2)}$ (see e.g. the discussion on [43]). Since we are adding to such a background a space-time filling D-brane wrapping an n-dimensional internal cycle, the $\text{SO}(1, 3)$ symmetry of the four-dimensional part is unbroken, while the two possible $\text{SU}(3)$ reductions of the internal $\text{SO}(6)$ structure group are generally spontaneously broken. Note that the chiral multiplets are the only ones who transform under the $\text{SU}(3)$ structure groups of the internal manifold, while the gauge multiplets are singlets with respect to it.

Turning to fermions, before fixing the κ-symmetry in the superaction of the D-brane, the fermionic content is given by a pair of ten dimensional Majorana-Weyl fermions $\theta^{(1)}$

and $\theta^{(2)}$ of opposite/same chirality in Type IIA/IIB, each transforming under one of the two SU(3)'s structure groups. The covariant κ-fixing used here consists in imposing the condition $\theta^{(2)} = 0$, keeping only $\theta \equiv \theta^{(1)}$ as physical degrees of freedom (for which the supersymmetry transformations take the form (3.9)). Note that this type of κ-fixing select the SU(3) associated to $\eta_+^{(1)}$, that we call SU(3)$_1$ from now on, as the natural one to be used to classify the world-volume fields. Of course, we could have made the alternative (still covariant) choice $\theta^{(1)} = 0$, but this would have been a little less natural since the physical fermion $\theta^{(2)}$ has different chirality for Type IIA and IIB, leading to a somehow less mirror-symmetric description. This natural selection of the SU(3)$_1$ structure group will emerge again in the following discussions.

Now, the sixteen components of the ten dimensional fermion θ splits in the following way under the full structure group $\mathrm{Spin}(1,3) \times \mathrm{SU}(3)_1$

$$\mathbf{16} \to (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \bar{\mathbf{3}}) + \mathrm{c.c.} \quad . \tag{3.11}$$

The four-dimensional vector fields transform as singlets under the internal SU(3)$_1$ structure group, while the scalar fields transform in the $\mathbf{3} + \bar{\mathbf{3}}$ representation. Then, the fermions in the $(\mathbf{2}, \mathbf{1})$ sector must be clearly included in the vector multiples and identified with the gaugini λ, while the $(\mathbf{2}, \bar{\mathbf{3}})$ sector is given by the fermionic fields χ^m of the chiral multiplets. Since a base for the $(\mathbf{2}, \mathbf{1})$ sector is given by $\zeta_+ \otimes \eta_+^{(1)}$, while a base for the $(\mathbf{2}, \bar{\mathbf{3}})$ sector is given by $\zeta_+ \otimes \hat{\gamma}_m \eta_-^{(1)}$, we can extract λ and χ^m from the following splitting of θ

$$\theta(x,\sigma) = e^{-2A(\sigma)} \lambda(x,\sigma) \otimes \eta_+^{(1)}(\sigma) + \frac{1}{\sqrt{2}} e^{-A(\sigma)} \chi^m(x,\sigma) \otimes \hat{\gamma}_m \eta_-^{(1)}(\sigma) + \mathrm{c.c.}, \tag{3.12}$$

where we have indicated explicitly the dependence on the flat and internal world-volume coordinates x^μ and σ^α. The normalizations in (3.12) have been fixed by requiring that λ and χ^m must have canonical kinetic term, using the internal metrics (3.5) and (3.6) for λ and χ^m respectively. Indeed, the explicit form of the quadratic fermionic terms on a general background was found in [44] and using this it is easy to see that the kinetic term for λ and χ^m are given by

$$\begin{aligned}
\mathcal{L}_{\mathrm{kin}}^F &= i \int_\Sigma \bar{\lambda} \gamma^\mu \partial_\mu \lambda P[e^{-\Phi} \mathrm{Re} \hat{\Psi}_1] \wedge e^{\mathcal{F}} + i \int_\Sigma g_{mn} \bar{\chi}^m \gamma^\mu \partial_\mu \chi^n P[e^{2A-\Phi} \mathrm{Re} \hat{\Psi}_1] \wedge e^{\mathcal{F}} \\
&= ik(\bar{\lambda}, \gamma^\mu \partial_\mu \lambda) + iG(\bar{\chi}, \gamma^\mu \partial_\mu \chi) \; .
\end{aligned} \tag{3.13}$$

Thus, from (3.10) we obtain the standard supersymmetry transformations (for constant field configurations)

$$\delta_\zeta \lambda = i \mathcal{D}^* \zeta \,, \qquad \delta_\zeta \chi^m = \sqrt{2} \, \mathcal{W}^{*m} \zeta \; . \tag{3.14}$$

Even if we have not computed precisely the dimensional reduction and appropriately organized all the tower of KK fields in supersymmetric multiplets, we can nevertheless conclude that from the four-dimensional point of view one can indeed consider $\mathcal{D}$ as a D-term and $\mathcal{W}_m$ as an F-term, motivating the names used to label the supersymmetry conditions (2.16).

Let us stress again that the above analysis was done in the 'linearized' approximation where $\mathcal{D}$ and $\mathcal{W}_m$ are small and we expect the theory to be well described by a standard

$\mathcal{N} = 1$ theory. Note that for the full DBI theory the vanishing of the D-term $\mathcal{D}$ alone is not enough to insure the vanishing of the gaugini supersymmetry transformations (see (3.9)). On the other hand, the vanishing of the F-term $\mathcal{W}_m$ alone *is* enough to insure that vanishing of supersymmetry transformation of the fermions in the chiral multiplets. We will see in the next section that the F-terms do indeed have a clear non-linear validity, by constructing the explicit complete superpotential generating them. A nonlinear interpretation of the D-term will arise in section 9.

4. Superpotential for D-branes on $\mathcal{N} = 1$ vacua

In section 2 we have recalled how supersymmetric D-branes in D-calibrated $\mathcal{N} = 1$ backgrounds can be seen as calibrated D-branes with respect to properly defined generalized calibrations. This condition is in turn equivalent (up to an appropriate orientation choice) to the pair of conditions given in (2.16) that, as discussed in section 3, may be seen as the F-flatness and D-flatness conditions in the language of the $\mathcal{N} = 1$ four-dimensional formulation. In this section we show how the F-flatness condition in (2.16) can be further extracted from a corresponding superpotential. This provides a generalization and, in some sense, a reformulation in a unified language, of previous superpotentials obtained in the Calabi-Yau case along the lines of the two-cycle case considered in [17] (see for example [22] for a general discussion in the Calabi-Yau case). Our superpotential is obviously applicable also in the limiting case of $\mathcal{N} = 2$ backgrounds with only nontrivial NS fields and their simplest subcase in which the internal manifold reduces to a standard Calabi-Yau.

Let us start by discussing the space of relevant degrees of freedom. We take as configuration space $\mathcal{C}$ the space of all the generalized cycles $(\Sigma, \mathcal{F})$ quotiented by the group of internal world-volume diffeomorphisms $\mathrm{Diff}(\Sigma)$. The space $\mathcal{C}$ can be properly identified with the space of the scalar fields in the four-dimensional description of the system. The world-volume gauge transformations that depends only on the Σ-coordinates (and not on the ones filling the four flat directions) corresponds to an infinite family of abelian *rigid* symmetries of the scalar field space $\mathcal{C}$, that are gauged in the full theory. The tangent space $T_{\mathcal{C}}$ to $\mathcal{C}$ should describe the infinitesimal deformations of the embedded submanifolds and of the world-volume field strength on them.

We first describe the deformations of the field strength $\mathcal{F}$ due to the deformations of the world-volume gauge field, while keeping the embedded submanifold Σ fixed. Since $\mathcal{F}$ must satisfy the generalized Bianchi identity $d\mathcal{F} = P_\Sigma[H]$, an infinitesimal variation of $\mathcal{F}$ must be of the form $\delta\mathcal{F} = da$, where $a \in \Gamma(T_\Sigma^*)$ is a globally defined one-form on Σ. As we have said, the infinitesimal gauge transformations $a \to a + d\lambda$, with $\lambda \in \Gamma(\Sigma, \mathbb{R})$, can be considered as the rigid transformations in the four-dimensional description of the system, that are gauged by the coupling to the four-dimensional vector fields. Secondly, we consider the general class of deformations of the submanifold Σ in M generated by a section $X \in \Gamma(T_M|_\Sigma)$ of the bundle T_M restricted to Σ. Note that such a a deformation induces also a corresponding infinitesimal transformation $\delta\mathcal{F} = P_\Sigma[\imath_X H]$ on the world-volume field-strength. A more detailed discussion on the infinitesimal deformations of the generalized cycle $(\Sigma, \mathcal{F})$ is contained in appendix A.

Now, not all these infinitesimal deformations are physically distinguishable since some could be related by an infinitesimal Σ-diffeomorphism. At the infinitesimal level, a Σ-diffeomorphism can be identified by a vector field $v \in \Gamma(T_\Sigma)$. Then, by associating v to its push-forward in $\Gamma(T_M|_\Sigma)$, we obtain that the infinitesimal transformations of the form

$$X = v, \qquad \delta\mathcal{F} = d\imath_v\mathcal{F} + P[\imath_v H], \tag{4.1}$$

must be considered as non-physical, and must be quotiented out. Then we must consider the following "gauge" equivalence between two infinitesimal deformations of the generalized cycle $(\Sigma, \mathcal{F})$

$$X \simeq X + v, \qquad a \simeq a + \imath_v\mathcal{F}. \tag{4.2}$$

Such identifications can be appropriately described in generalized geometry terms, by recalling the definition of generalized tangent bundle $\tau_{(\Sigma,\mathcal{F})}$ of a generalized cycle $(\Sigma, \mathcal{F})$ given in [7]:

$$\tau_{(\Sigma,\mathcal{F})} = \{v + \eta \in T_\Sigma \oplus T_M^*|_\Sigma \ : \ P_\Sigma[\eta] = \imath_v\mathcal{F}\}. \tag{4.3}$$

From (4.3) it is clear that the tangent space $T_\mathcal{C}$ of $\mathcal{C}$ at a "point" $(\Sigma, \mathcal{F})$ can be identified with the space of sections of the vector bundle $\mathcal{N}_{(\Sigma,\mathcal{F})} \equiv (T_M \oplus T_M^*)|_\Sigma / \tau_{(\Sigma,\mathcal{F})}$, that we call the *generalized normal bundle* of $(\Sigma, \mathcal{F})$.

We are now ready to rewrite the F-flatness condition in (2.16) in a form that can be more immediately recognized as coming from a superpotential. In order to do this, let us start by splitting the F-flatness condition in two, by projecting it in the orthogonal and tangent directions using the metric structure of the background. We consider first an arbitrary vector field $X_\perp \in T_\Sigma^\perp$ orthogonal to Σ. If we consider the projection of the F-flatness condition in (2.16) along $X_\perp$, we obtain

$$P_\Sigma[\imath_{X_\perp}\hat\Psi_2] \wedge e^\mathcal{F}|_{\text{top}} = 0. \tag{4.4}$$

Secondly, we consider an arbitrary section $X_{||}$ of the tangent bundle T_Σ of Σ. Projecting the F-flatness condition along $X_{||}$ gives the equation

$$P[X_{||}^* \wedge \hat\Psi_2 + \imath_{X_{||}}\hat\Psi_2] \wedge e^\mathcal{F}|_{\text{top}} = 0, \tag{4.5}$$

where $X_{||}^*$ is the one-form canonically associated to $X_{||}$ through the world-volume metric $P[g]$. It is easy to see that it is possible to write this equation in the following equivalent way (we use internal world-volume coordinates σ^α)

$$(P[g] + \mathcal{F})_{\alpha\beta} X_{||}^\beta d\sigma^\alpha \wedge P[\hat\Psi_2] \wedge e^\mathcal{F} = 0. \tag{4.6}$$

Thus, since $P[g] + \mathcal{F}$ is non-degenerate for non-degenerate brane configurations, we can rewrite the F-flatness condition in (2.16) as the following pair of conditions

$$\begin{aligned} P_\Sigma[\hat\Psi_2] \wedge e^\mathcal{F}|_{\text{top}-1} &= 0, \\ P_\Sigma[\imath_X\hat\Psi_2] \wedge e^\mathcal{F}|_{\text{top}} &= 0, \end{aligned} \tag{4.7}$$

where now X is an arbitrary section of $T_M|_\Sigma$. Note that, by using the first of the F-flatness conditions (4.7), the second can in fact be though as X was actually a section the canonical normal bundle $\mathcal{N}_\Sigma = T_M/T_\Sigma$, since it is left invariant if we substitute X with $X + v$ for any $v \in \Gamma(T_\Sigma)$.

We can now present the superpotential generating the F-flatness conditions (4.7), postponing to section 7 the discussion of how it can be actually considered as holomorphic. We want to define a superpotential $\mathcal{W}$ as functional of the pair $(\Sigma, \mathcal{F})$ defining the internal configuration of the four-dimensional space-time filling D-brane. In order to define such a functional, we need to introduce a fixed reference generalized cycle $(\Sigma_0, \mathcal{F}_0)$ which is smoothly related to $(\Sigma, \mathcal{F})$. More precisely, we require that $(\Sigma_0, \mathcal{F}_0)$ is in the same *generalized homology class* of $(\Sigma, \mathcal{F})$, that is there must exist a chain $\mathcal{B}$ and a field strength $\tilde{\mathcal{F}}$ on it (satisfying $d\tilde{\mathcal{F}} = P_\mathcal{B}[H]$) such that[8]

$$\partial\mathcal{B} = \Sigma - \Sigma_0 , \qquad P_\Sigma[\tilde{\mathcal{F}}] = \mathcal{F} \quad \text{and} \quad P_{\Sigma_0}[\tilde{\mathcal{F}}] = \mathcal{F}_0 . \tag{4.8}$$

Then the superpotential whose critical points are given by the F-flatness conditions (4.7) can be defined by

$$\mathcal{W}(\Sigma, \mathcal{F}) - \mathcal{W}(\Sigma_0, \mathcal{F}_0) = \frac{1}{2} \int_\mathcal{B} P[e^{3A-\Phi}\hat{\Psi}_2] \wedge e^{\tilde{\mathcal{F}}} . \tag{4.9}$$

The formula (4.9) defines the superpotential $\mathcal{W}(\Sigma, \mathcal{F})$ up to an additive constant, whose indeterminacy comes from the arbitrary choice of $(\Sigma_0, \mathcal{F}_0)$ and also by the possible nontrivial topology of the background.[9] We will see in the next section how we can give to $\mathcal{B}$ and $\tilde{\mathcal{F}}$ defined in (4.8) a clear physical interpretation.

It is immediate to obtain (4.7) as critical point conditions for the superpotential (4.9). Indeed, consider any generalized normal vector $[(X, a)] \in \Gamma(\mathcal{N}_{(\Sigma, \mathcal{F})})$, associated to the representative (X, a). Then, the infinitesimal variation of $\mathcal{W}$ defined by $[(X, a)]$ is given by

$$\delta\mathcal{W} = \frac{1}{2} \int_\Sigma \left\{ P[e^{3A-\Phi}\imath_X\hat{\Psi}_2] + a \wedge P[e^{3A-\Phi}\hat{\Psi}_2] \right\} \wedge e^\mathcal{F} . \tag{4.10}$$

Note that clearly the above infinitesimal variation is invariant under the substitution $(X, a) \to (X + v, a + \imath_v\mathcal{F})$, for any $v \in \Gamma(T_\Sigma)$, and thus it is well defined for the equivalence class $[(X, a)]$. From (4.10) it is clear that the superpotential critical points are defined by the conditions (4.7). Note also that the two terms (4.10) can be directly identified with the left hand side of (4.4) and (4.6) by choosing a gauge with $X = X_\perp$ orthogonal to Σ and making the identification

$$a = (P[g] + \mathcal{F}) \cdot X_{||} . \tag{4.11}$$

---

[8]In the case in which Σ has zero homology class we can take an empty Σ_0 and the conditions (4.8) can be simplified to the pair of conditions $\partial\mathcal{B} = \Sigma$ and $P_\Sigma[\tilde{\mathcal{F}}] = \mathcal{F}$.

[9]If for example the homology group $H_{\dim\Sigma+1}(M, \mathbb{R})$ is non-zero, there are possible non-homologous choices of $\mathcal{B}$ (for fixed boundary conditions). The choice of a different class in $H_{\dim\Sigma+1}(M, \mathbb{R})$ gives a shift of $\mathcal{W}$ by a constant.

This provides an explicit identification of $T_{\mathcal{C}}|_{(\Sigma,\mathcal{F})} = \Gamma(\mathcal{N}_{(\Sigma,\mathcal{F})})$ with $\Gamma(T_M|_\Sigma)$, which uses in an essential way the background metric. In the following we will often use this identification, which will allow us to introduce an almost complex and a symplectic structure on $\mathcal{C}$.

Note that, as the F-flatness conditions in the form (4.7), the superpotential does not depend on the full $SU(3) \times SU(3)$ structure (which contains also the metric structure) characterizing the internal manifold of the $\mathcal{N} = 1$ backgrounds we are considering, but it involves only the integrable pure spinor. This result could be seen as a generalization of the "decoupling statement" presented in [10], which asserts that the superpotentials governing D-branes in Calabi-Yau spaces depend only on the background complex structure and not on the Kähler structure for B-branes, and vice-versa for A-branes. The same superpotential may be used to describe also topological D-branes $[11-15]$ for the underlying topological model [11, 16, 5], since its form is clearly valid for any generalized Calabi-Yau structure, as defined by Hitchin in [6]. Namely, for any generalized Calabi-Yau manifold defined by a d_H-closed pure spinor ψ, we can introduce a variational problem to characterize the generalized complex submanifolds $(\Sigma, \mathcal{F})$ as the extrema of the functional

$$F(\Sigma, \mathcal{F}) = \frac{1}{2} \int_{\mathcal{B}} P[\psi] \wedge e^{\tilde{\mathcal{F}}} \,, \tag{4.12}$$

where $\mathcal{B}$ and $\tilde{\mathcal{F}}$ are defined as for the specific case of the $\mathcal{N} = 1$ backgrounds considered before.

The above superpotentials can be written directly in terms of the generalized cycle $(\Sigma, \mathcal{F})$ by using the d_H-closedness of $e^{3A-\Phi}\hat{\Psi}_2$ (or analogously of ψ in (4.12)). Indeed, we can locally write $e^{3A-\Phi}\hat{\Psi}_2 = d_H\chi$, where χ is again a polyform, and then

$$\mathcal{W}(\Sigma, \mathcal{F}) = \frac{1}{2} \int_{\Sigma} P[\chi] \wedge e^{\mathcal{F}} + \text{constant} \,. \tag{4.13}$$

Note that the expression (4.9) for the superpotential is completely analogous to the CS term of the D-brane action and like that it is meaningful even if the 'potential' polyform χ is not generally globally defined.

To close this section, let us stress that till now we have deliberately ignored the tension $\mu_p = 2\pi(2\pi\sqrt{\alpha'})^{-(p+1)}g_s^{-1}$ of the Dp-brane we are considering (i.e., we have fixed $\mu_p = 1$). The tension should be of course reintroduced to have the correct dependence on the fundamental quantities α' and g_s. The canonically normalized superpotential $\mathcal{W}_{\text{can}}$ which includes the correct dependence on the tension is given by

$$\mathcal{W}_{\text{can}} = \mu_p \mathcal{W} \,, \tag{4.14}$$

as follows directly from the form of the potential (3.8), since the canonically normalized potential and metric are given by $\mathcal{V}_{\text{can}} = \mu_p \mathcal{V}$ and $G_{\text{can}} = \mu_p G$ respectively, where G is defined in (3.6). As we will see in the following section, the superpotential (4.9) can be derived from an argument involving domain walls, which also gives an alternative consistency check of the above normalization of the superpotential. In the following we will re-intruduce the correct dependence on the tension only when needed, continuing to neglect it in most of the discussions.

5. Superpotential from domain walls

In the previous section we have shown how to obtain the F-flatness conditions in (2.16) or equivalently (4.7) as the conditions defining the critical points of the superpotential (4.9). In this section we use a physical argument that leads directly to the above superpotential, confirming its validity from a more physical point of view. This can be seen as a generalization to the D-brane context of the standard Gukov-Vafa-Witten argument used to derive the superpotential governing supergravity compactifications with fluxes [24, 25]. In particular we will see how the domain wall generalized calibration given in (2.15), being naturally related to the integrable generalized complex structure of the background, is also naturally related to the F-term associated to the space-time filling D-brane. Along the way, it will allow to check the canonical normalization of the superpotential given in (4.14).

For a given space-time filling D-brane, consider two supersymmetric configurations $(\Sigma_1, \mathcal{F}_1)$ and $(\Sigma_2, \mathcal{F}_2)$ that belong to the same generalized homology class. These can be seen as $\mathcal{N} = 1$ vacua of the effective $\mathcal{N} = 1$ four-dimensional supersymmetric theory governing the D-brane dynamics. Then, on general grounds, we expect that a domain wall interpolating between the two vacua can exist. Such a domain wall configuration can be constructed in the following way. Take a D-brane filling the half of space-time with positive third space coordinate, $x^3 > 0$, and wrapping the supersymmetric generalized cycle $(\Sigma_1, \mathcal{F}_1)$, and another D-brane (of the same kind) filling the other half of space-time with $x^3 < 0$ and wrapping the other supersymmetric generalized cycle $(\Sigma_2, \mathcal{F}_2)$. These two D-brane configurations with boundary $\mathbb{R}^{1,2} \times \{x^3 = 0\}$ can be glued together in a consistent way by filling the common boundary with another D-brane (again, of the same kind) wrapping a generalized cycle $(\mathcal{B}, \tilde{\mathcal{F}})$ defined by a chain $\mathcal{B}$ with boundary such that $\partial \mathcal{B} = \Sigma_1 - \Sigma_2$ and a world-volume field-strength $\tilde{\mathcal{F}}$ such that $P_{\Sigma_1}[\tilde{\mathcal{F}}] = \mathcal{F}_1$ and $P_{\Sigma_2}[\tilde{\mathcal{F}}] = \mathcal{F}_2$. The choice of the field-strength $\tilde{\mathcal{F}}$ is the right one to glue together the three D-brane configurations with boundaries in such a way that the usual anomaly terms coming from the boundaries of each D-brane [45 – 47] cancel each other. In order to see it, let us write the complete set of Ramond-Ramond potentials in the form $C = \sum_k C_{(k)}$, where k is odd in Type IIA and even in Type IIB, and consider the general gauge transformation $\delta C = d_H \lambda$, where $\lambda = \sum_k \lambda_{(k-1)}$. The CS term in the action of the two half space-time filling D-branes transforms in the following way

$$
\begin{aligned}
\delta S_1^{\mathrm{CS}} + \delta S_2^{\mathrm{CS}} &= \delta \int_{\mathbb{R}^{1,2} \times \mathbb{R}_+ \times \Sigma_1} P[C] \wedge e^{\mathcal{F}_1} + \delta \int_{\mathbb{R}^{1,2} \times \mathbb{R}_- \times \Sigma_2} P[C] \wedge e^{\mathcal{F}_2} \\
&= - \int_{\mathbb{R}^{1,2} \times \Sigma_1} P[\lambda] \wedge e^{\mathcal{F}_1} + \int_{\mathbb{R}^{1,2} \times \Sigma_2} P[\lambda] \wedge e^{\mathcal{F}_2} \ .
\end{aligned}
\tag{5.1}
$$

Then the gauge symmetry is broken by the boundary terms if we consider the two half space-time filling D-branes alone. However, the introduction of the domain wall D-brane located at $x^3 = 0$ as described above provides the necessary counterterm to reabsorb the undesired terms in (5.1). Indeed, the domain wall D-brane action contains the CS term

$$
S_{\mathrm{DW}}^{\mathrm{CS}} = \int_{\mathbb{R}^{1,2} \times \mathcal{B}} P[C] \wedge e^{\tilde{\mathcal{F}}} \ ,
\tag{5.2}
$$

and it is easy to see that its variation under the gauge transformation $d_H\lambda$ exactly cancels the two terms in (5.1).

Now, from general arguments in $\mathcal{N} = 1$ supersymmetric field theories (see e.g. [48]), it is known that the tension of a BPS domain wall is simply given by

$$T_{\mathrm{DW}} = 2\mathrm{Re}(e^{i\theta}\Delta\mathcal{W})\,, \tag{5.3}$$

where $\Delta\mathcal{W} = \mathcal{W}_1 - \mathcal{W}_2$ is the superpotential difference of the two different vacua and θ define a constant phase related to the preserved half of supersymmetry. On the other hand, from our D-brane construction the field theory domain wall tension should be exactly given by the effective tension of a supersymmetric configuration for the D-brane domain wall introduced above. But, from the general discussion of [8] reviewed in section 2, we know that such a supersymmetric domain wall D-brane must wrap a generalized cycle calibrated with respect to the generalized calibration $\omega^{(\mathrm{DW})}$ written in (2.15). From this, we immediately obtain that the tension of the BPS D-brane domain wall is given by

$$T_{\mathrm{DW}} = \int_{\mathcal{B}} P[e^{3A-\Phi}\mathrm{Re}(e^{i\theta}\hat{\Psi}_2)] \wedge e^{\tilde{\mathcal{F}}}\,, \tag{5.4}$$

where again θ defines the preserved supersymmetry. Comparing this expression with the one given in (5.3), one can immediately extract the the form of the superpotential as written in (4.9) (again defined up to an additive constant). Furthermore, by reintroducing the neglected tension μ_p in front of the right hand side of (5.4), we obtain the canonically normalized superpotential (4.14). Note that, from the general analysis of [8], the fact that the domain wall D-brane is calibrated with respect to the generalized calibration $\omega^{(\mathrm{DW})}$ of (2.15) implies also that $P_{\mathcal{B}}[e^{3A-\Phi}\mathrm{Im}(e^{i\theta}\hat{\Psi}_2)] \wedge e^{\tilde{\mathcal{F}}}|_{\mathrm{top}} = 0$. Thus, as in field theory, the phase θ in (5.4) is directly related to the phase of superpotential difference, i.e. $e^{-i\theta} = \Delta\mathcal{W}/|\Delta\mathcal{W}|$, so that

$$T_{\mathrm{DW}} = 2|\Delta\mathcal{W}| = \left|\int_{\mathcal{B}} P[e^{3A-\Phi}\hat{\Psi}_2] \wedge e^{\tilde{\mathcal{F}}}\right|\,. \tag{5.5}$$

6. Fayet-Iliopoulos terms and cosmic strings

In the previous section we have seen how the well known relation between the superpotential of an $\mathcal{N} = 1$ theory and supersymmetric domain walls can be exactly reproduced in our D-brane context by using the calibration $\omega^{(\mathrm{DW})}$ defined in (2.15) for D-branes filling only three flat space-time directions.

It this section we will discuss how, on the other hand, our D-terms are related to the another possible solitonic objects allowed by an $\mathcal{N} = 1$ theory, namely cosmic strings.[10] There has been a lot of recent activity focused on the embedding of these kind of solitons into string theory (for a review see for example [49]). In particular, in [27] it has been

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[10]Using this name, we implicitly refer to cosmological scenarios obtained from flux compactifications. In the context of the gauge/gravity correspondence, these effective string configurations can be also seen as proper solitonic objects of rigid supersymmetric theories.

stressed how the only allowed supersymmetric cosmic string solutions of four-dimensional $\mathcal{N} = 1$ supergravity must have a vanishing F-term and can exist thanks to D-terms with a non-vanishing constant Fayet-Iliopoulos (FI) term. Furthermore the authors of [27] proposed an identification of the $\mathcal{N} = 1$ four-dimensional supergravity they started from with the effective supergravity theory describing some main features of a D$\bar{\text{D}}$-brane pair filling the four flat space-time dimensions and wrapping some internal cycle (see also the related discussions in [50, 51]). Our formalism allows to give a non-trivial explicit argument in favor of this proposal and a direct D-brane derivation of some of the results of [27] (see also [53, 52]).

Let us start by considering a single space-time filling Dp-brane wrapping an internal n-dimensional generalized cycle $(\Sigma, \mathcal{F})$. The crucial observation is that the D-flatness condition $\mathcal{D}(\Sigma, \mathcal{F}) = 0$ [the D-term $\mathcal{D}$ is defined in (3.1)] can be satisfied only if $\int_\Sigma \mathcal{D} d^n \sigma = 0$. By recognizing in $\mathcal{D}$ the presence of the string generalized calibration $\omega^{(\text{string})}$ written in (2.15), which is d_H-closed, we immediately see that this condition is topological, i.e. does not change if we continuously deform $(\Sigma, \mathcal{F})$. Then, from the analysis of section 3, it is natural to identify the constant (reintroducing the tension of the D-brane)

$$\xi \equiv \mu_p \int_\Sigma \mathcal{D} d^n \sigma \qquad (6.1)$$

with the FI term of the lowest Kaluza-Klein four-dimensional U(1) gauge field. Indeed, the corresponding gauge group has no associated charged chiral fields and thus the necessary requirement for having a supersymmetric vacuum is that $\xi = 0$. Note that even if $\mathcal{D}$ was identified as a D-term expanding the action around a supersymmetric configuration, the fact that ξ defined in (6.1) is constant for *any* configuration supports the idea that its identification with an effective FI term should indeed be more general. This will be confirmed by the following analysis.

Take a space-time filling Dp-brane wrapping a generalized cycle $(\Sigma, \mathcal{F})$ such that $\xi \neq 0$. As we have said, this cannot admit a supersymmetric configuration (at least considering only classical deformations). However, we can add an anti $\bar{\text{D}}p$-brane wrapping the same internal generalized cycle $(\Sigma, \mathcal{F})$. As a consequence, the resulting spectrum on the branes includes now also a complex tachyon which is charged under the combination $A^{(1)} - A^{(2)}$ of the two gauge fields $A^{(1)}$ and $A^{(2)}$ living on the two branes. Thus, from the discussion of the previous paragraph, it seems reasonable to conclude that the lowest Kaluza-Klein mode of the diagonal U(1) gauge group under which the tachyon is charged has ξ as non-vanishing FI term. The (unstable) system then admits a vortex solution [54] that can be identified with a D$(p-2)$-brane filling only two flat space-time directions and wrapping the internal $(\Sigma, \mathcal{F})$-cycle, thus leaving an effective cosmic string. From the analysis of [8], we can immediately conclude that the resulting cosmic string is supersymmetric if and only if it is calibrated with respect to the generalized calibration $\omega^{(\text{string})}$. This implies that the cosmic sting tension is given by

$$T_{\text{string}} = \mu_{p-2} \int_\Sigma \omega^{(\text{string})} \wedge e^{\mathcal{F}} = (2\pi)^2 \alpha' \xi . \qquad (6.2)$$

On the other hand, since we are considering $\mathcal{N} = 1$ backgrounds, the D$\bar{\text{D}}$-brane system should be described by a four-dimensional $\mathcal{N} = 1$ low energy effective theory. Moreover, since we consider BPS cosmic strings, their tension computed in (6.2) using a probe D$(p-2)$-brane should be reproduced by the four-dimensional results of [27]. Indeed, to recognize the perfect agreement it is enough to remember that in the description given in section 3 we have used fluctuating fields with the dimension of a length. The standard dimensions for the fields are obtained by simply rescaling them by $2\pi\alpha'$. This induces a corresponding rescaling $\xi \to \xi/2\pi\alpha'$ of the FI term. Thus, in terms of the proper dimensional FI term, the cosmic string tension reads $T_{\text{string}} = 2\pi\xi$, which is exactly reproduced by the effective supergravity calculation of [27].

Our argument also allows one to obtain from a purely D-brane setting the observation of [27] that for BPS cosmic strings of an $\mathcal{N} = 1$ four-dimensional supergravity the F-term must vanish identically. Indeed, from the discussion of [8] it follows that the calibration condition on the generalized cycle $(\Sigma, \mathcal{F})$ wrapped by the D-brane forming a BPS cosmic string implies also that $(\Sigma, \mathcal{F})$ must be a generalized complex submanifold, i.e. the F-term must vanish identically so that the superpotential (4.9) is extremized everywhere.

Let us stress another outcome of our approach. The system constituted by a D$\bar{\text{D}}$-brane pair added to an $\mathcal{N} = 1$ background should be described by an effective $\mathcal{N} = 1$ supergravity theory like the one considered in [27]. As is clear from the above analysis, we can obtain an effective cosmic string as a tachyonic vortex on a D$\bar{\text{D}}$-brane pair only if these space-time filling branes wrap an internal generalized cycle $(\Sigma, \mathcal{F})$ that *cannot* be deformed in such a way that the two D-branes, taken singularly, become supersymmetric. In few words, we must start from a pair of non-supersymmetric space-time filling D-branes if we want to create a cosmic string from tachyon condensation. Vice-versa, if we start from supersymmetric Dp-branes, then tachyon condensation cannot give rise to any supersymmetric D$(p-2)$-brane configuration wrapping a generalized cycle homologous to $(\Sigma, \mathcal{F})$. This is an immediate consequence of the fact that in general parallel Dp- and D$(p-2)$-branes do not separately preserve any common supersymmetry (but generally form a proper bound state). In our case, the $\mathcal{N} = 1$ supersymmetry of the background implies that a generalized n-cycle cannot be contemporary homologous to generalized calibrated cycles for both D$(3+n)$- and D$(1+n)$-branes.

This last conclusion cannot be extended to the particular subcases where the RR fields are switched off and the background preserves $\mathcal{N} = 2$ supersymmetry. Indeed, in these cases we have an arbitrary phase entering the generalized calibrations (that can be adjusted giving a different preserved internal supersymmetry) and the condition for a generalized cycle to be calibrated does not depend on the number of filled flat directions [38, 9]. However, a supersymmetric D$(1+n)$-brane wrapping a generalized n-cycle preserves exactly the $\mathcal{N} = 1$ supersymmetry that is broken by a D$(3+n)$-brane wrapping the same generalized n-cycle. The associated non-linearly realized supersymmetry on the world-volume of the D$(3+n)$-branes constituting the D$\bar{\text{D}}$-brane pair should then be associated to a FI term ξ in a four-dimensional $\mathcal{N} = 1$ description of the system, as happens for $\mathcal{N} = 1$ backgrounds. Then, the above analysis for $\mathcal{N} = 1$ backgrounds can be repeated with no changes giving again $T_{\text{string}} = 2\pi\xi$. It would be interesting to understand better the relation between

the D-brane picture and a complete $\mathcal{N} = 2$ four-dimensional supergravity description of one-half BPS cosmic strings, like for example the one presented in [55].

7. Holomorphicity of the superpotential

We can now pass to the discussion of the holomorphic structure of the superpotential introduced in section 4. More precisely, we will introduce an almost complex structure on the space $\mathcal{C}$ of the generalized cycles $(\Sigma, \mathcal{F})$ with respect to which the superpotential is holomorphic, i.e. it is annihilated by the $(0, 1)$ vectors on $\mathcal{C}$. Since the space of possible deformations is infinite dimensional, we will work quite at the formal level treating it as finite dimensional, neglecting possible related subtleties. Furthermore, we shall not worry about the integrability of the almost complex structures introduced. Such an issue is already present for example in the study of Lagrangian submanifolds [22], but is not so crucial for the following discussion.

Let us start by recalling that the internal manifold M has an integrable generalized complex structure $\mathcal{J}_2$ associated to the integrable pure spinor Ψ_2. It is clearly not sufficient by itself to induce an almost complex structure (integrable or not) on $\mathcal{C}$. However, it *does* define a natural almost complex structure, in the sense of an endomorphism of the tangent bundle that squares to minus one, if we restrict $T_{\mathcal{C}}$ to the subspace $\mathcal{C}_{\mathrm{hol}} \subset \mathcal{C}$ of the generalized complex submanifolds. As we have seen in section 3, $\mathcal{C}_{\mathrm{hol}}$ can be characterized as the space of critical points of the superpotential (4.9). Indeed, by definition a generalized cycle $(\Sigma, \mathcal{F})$ is complex if the associated tangent bundle $\tau_{(\Sigma, \mathcal{F})}$ is stable under $\mathcal{J}_2$. As a consequence, $\mathcal{J}_2$ defines a natural almost complex structure on the generalized normal bundle $\mathcal{N}_{(\Sigma, \mathcal{F})}$ and then on the subset $\mathcal{C}_{\mathrm{hol}}$ of $\mathcal{C}$ using the identification $T_{\mathcal{C}}|_{(\Sigma, \mathcal{F})} = \Gamma(\mathcal{N}_{(\Sigma, \mathcal{F})})$.

Now, we would like to introduce an appropriate (almost) complex structure $\mathbb{J}$ on $T_{\mathcal{C}}$ that should provide an extension to the whole $\mathcal{C}$ of the complex structure properly defined only on $\mathcal{C}_{\mathrm{hol}}$. Furthermore, the metric introduced in (3.6) will turn out to define an associated Hermitian metric on $\mathcal{C}$. In order to do it we must first of all use the generalized metric structure [7] on $T_M \oplus T_M^*$, that in our case is ultimately given by the metric g of M, to find a good coordinatization of $\mathcal{N}_{(\Sigma, \mathcal{F})}$. Using the metric g we can split $T_M|_{\Sigma}$ in the sum of the tangent and orthogonal bundles to Σ, $T_M|_{\Sigma} = T_{\Sigma} \oplus T_{\Sigma}^{\perp}$. Then, we can give a global of splitting of $T_M \oplus T_M^*|_{\Sigma}$ appearing in the short exact sequence

$$0 \rightarrow \tau_{(\Sigma, \mathcal{F})} \rightarrow T_M \oplus T_M^*|_{\Sigma} \rightarrow \mathcal{N}_{(\Sigma, \mathcal{F})} \rightarrow 0 \,, \tag{7.1}$$

by using the Σ-diffeomorphism invariance to select $X = X_{\perp} \in T_{\Sigma}^{\perp}$ in the equivalence class $[(X, a)] \in \mathcal{N}_{(\Sigma, \mathcal{F})}$. This allows to identify $\mathcal{N}_{(\Sigma, \mathcal{F})}$ with $T_{\Sigma}^{\perp} \oplus T_{\Sigma}^*$ and then a general tangent vector of $T_{\mathcal{C}}|_{(\Sigma, \mathcal{F})}$ can be identified by a pair $(X_{\perp}, a) \in \Gamma(T_{\Sigma}^{\perp} \oplus T_{\Sigma}^*)$. We can also see this vector as a vector field $X = X_{\parallel} + X_{\perp} \in \Gamma(T_M|_{\Sigma})$, using the identification $a = (P[g] + \mathcal{F}) \cdot X_{\parallel}$ already introduced in (4.11) to relate the variation of the superpotential to the form of the F-flatness written in (2.16). At this point we must recall that for our $\mathcal{N} = 1$ backgrounds one can use the internal spinors $\eta_+^{(1)}$ and $\eta_+^{(2)}$ to construct a pair of almost complex structures $(J_1)_m{}^n = -(i/|a|^2)\eta_+^{(1)\dagger}\hat{\gamma}_m{}^n\eta_+^{(1)}$ and $(J_2)_m{}^n = -(i/|a|^2)\eta_+^{(2)\dagger}\hat{\gamma}_m{}^n\eta_+^{(2)}$ on M (see [28, 5], and [8] for the conventions used here). Moreover the internal metric g_{mn} is

Hermitian with respect to both of them. These almost complex structures define also the null spaces of the two pure spinors $\Psi^\pm$ since

$$(1 + iJ_1)_m{}^n(\imath_n + g_{nk}dy^k\wedge)\Psi^\pm = 0, \qquad (1 \mp iJ_2)_m{}^n(\imath_n - g_{nk}dy^k\wedge)\Psi^\pm = 0 . \qquad (7.2)$$

Note that J_1 is somehow selected by the property that its $+i$ eigenspace defines through the above equations the (complex) three dimensional space given by the intersection of the two null subspaces of the two pure spinors $\Psi^\pm$. Indeed, by looking at the F-flatness conditions as written in (2.16), it is clear that J_1 plays a particular role. We are then naturally led to use J_1 to define an almost complex structure on $T_M|_\Sigma$, and consequently obtain the almost complex structure $\mathbb{J}$ on $\mathcal{C}$ through the above identifications. Holomorphic and antiholomorphic tangent vectors in $T_{\mathcal{C}}|_{(\Sigma,\mathcal{F})}$ are given by (complex) vector fields Z and $\bar{Z}$, sections of $T_M^{\mathbb{C}}|_\Sigma$, satisfying the conditions $Z^m = \frac{1}{2}(1-iJ_1)_n{}^m Z^n$ and $\bar{Z}^m = \frac{1}{2}(1+iJ_1)_n{}^m \bar{Z}^n$ respectively. From (7.2) and the discussion of section 4 it is clear that the variation of a superpotential with respect to an anti-holomorphic $\bar{Z}$ vanish identically:

$$\bar{Z}(\mathcal{W}) = \frac{1}{2}\int \bar{Z}^m P[(\imath_m + g_{mk}dy^k\wedge)\hat{\Psi}_2] \wedge e^{\mathcal{F}} \equiv 0 . \qquad (7.3)$$

Then, the superpotential is holomorphic with respect to the almost complex structure $\mathbb{J}$. Furthermore, it clear that the metric G defined in (3.6) can be identified as a Hermitian metric on $\mathcal{C}$ naturally inherited from the background metric.

We would like now to argue that, if we restrict to $\mathcal{C}_{\text{hol}} \subset \mathcal{C}$, the almost complex structure $\mathbb{J}$ reduces to the one naturally induced by the integrable generalized complex structure $\mathcal{J}_2$ as described above. This can be understood when we give an interpretation of $\mathbb{J}$ from the point of view of the generalized complex geometry of the internal space M. Suppose to have a subbundle of $(T_M \oplus T_M^*)|_\Sigma$ that can be identified with $\mathcal{N}_{(\Sigma,\mathcal{F})}$ in a particular "gauge". If this subbundle is stable under the action of $\mathcal{J}_2$ then $\mathcal{J}_2$ can be used to define an almost complex structure on it. Thus $\mathcal{J}_2$ induces an almost complex structure on $\mathcal{N}_{(\Sigma,\mathcal{F})}$ and as a consequence on $\mathcal{C}$. The generalized metric structure given by the $\text{SU}(3) \times \text{SU}(3)$ structure of our backgrounds provides such a subbundle. Let us start by defining the following orthogonal subspaces of $T_M \oplus T_M^*$ [7]

$$C_\pm = \text{graph}\{\pm g : T_M \to T_M^*\} . \qquad (7.4)$$

Note that $C_+ \oplus C_- = T_M \oplus T_M^*$ and that $C_\pm$ are both isomorphic to T_M through the projection map $\pi : T_M \oplus T_M^* \to T_M$. Both $C_+|_\Sigma$ and $C_-|_\Sigma$ indeed provide a subbundle of $(T_M \oplus T_M^*)|_\Sigma$ that is isomorphic to $\mathcal{N}_{(\Sigma,\mathcal{F})}$. In order for them to be suitable for defining an almost complex structure on $\mathcal{N}_{(\Sigma,\mathcal{F})}$, and then on $\mathcal{C}$, we have to verify that $C_+|_\Sigma$ and $C_-|_\Sigma$ are stable under the action of $\mathcal{J}_2$. This can be seen by observing that, in our $\text{SU}(3) \times \text{SU}(3)$ structure manifolds, the integrable generalized complex structure $\mathcal{J}_2$ (and also the non-integrable $\mathcal{J}_1$) can be written in terms of J_1 and J_2 by restricting to C_+ and C_- and then using the isomorphism $C_\pm \simeq T_M$ [7]. More precisely, remembering that $\mathcal{J}_2$ is given by $\mathcal{J}_+$ in Type IIA and $\mathcal{J}_-$ in Type IIB, we have that

$$\mathcal{J}_\pm = \pi|_{C_+}^{-1} J_1 \pi P_+ \mp \pi|_{C_-}^{-1} J_2 \pi P_- , \qquad (7.5)$$

where $P_\pm$ are the projectors on $C_\pm$.

It is then clear that $C_\pm$ are stable under $\mathcal{J}_2$ and can be used to define an almost complex structure on $\mathcal{C}$ as explained above. In particular, using the isomorphism $C_\pm \simeq T_M$, the resulting almost complex structure is essentially given by J_1 if we use C_+ and by $-J_2$ or J_2, in Type IIA or Type IIB respectively, if we use C_-. We then see that the choice of C_+ is somehow selected by its invariance under mirror symmetry. Indeed, the resulting almost complex structure coincides with the one constructed previously in a more direct way, with respect to which the superpotential is holomorphic. Finally, note that in general the almost complex structure $\mathbb{J}$ defined in this way depends on the $SU(3) \times SU(3)$ structure of the background. However, when we restrict to $\mathcal{C}_{\mathrm{hol}}$, i.e. to generalized complex submanifolds, this obviously coincides with the natural one that, as we have already said, in this case can be defined referring only to the generalized complex structure $\mathcal{J}_2$.

Consider now the alternative subbundle of $(T_M \oplus T_M^*)|_\Sigma$ isomorphic to $\mathcal{N}_{(\Sigma,\mathcal{F})}$, whose elements are restricted to be of the form $(X_\perp, a) \in \Gamma(T_\Sigma^\perp \oplus T_\Sigma^*)$. Of course any element of $C_+|_\Sigma$ can be put in this form by an appropriate 'gauge' transformation. If $(X, g \cdot X) \in C_+$, we can identify it with its image under the translation given by $(-X_{||}, -\imath_{X_{||}}\mathcal{F} - g \cdot X_\perp)$ (where the meaning of the notation should be obvious). The resulting vector is given by $(X_\perp, (g + \mathcal{F}) \cdot X_{||})$. Then, using the isomorphism given by π to identify $(X, g \cdot X)$ with X, we also find an interpretation from the generalized complex geometry point of view of the identification

$$X = X_{||} + X_\perp \quad \leftrightarrow \quad (X_\perp, a) \quad \text{with} \quad a = (g + \mathcal{F}) \cdot X_{||} . \tag{7.6}$$

This identification was already introduced somehow ad hoc in section 4 to identify the F-flatness conditions in the form given in (2.16) as conditions for the critical points of the superpotential (4.9).

To summarize, we have constructed an almost complex structure $\mathbb{J}$ on the space $\mathcal{C}$ of the generalized cycles $(\Sigma, \mathcal{F})$, with respect to which the superpotential $\mathcal{W}$ defined in (4.9) is holomorphic and the metric G defined in (3.6) is Hermitian. At the end of the following section we will see how this almost complex structure induces also an almost complex structure on the space $\mathcal{C}_{\mathrm{hol}}$ that actually depends only on the integrable generalized complex structure on M.

8. Reduced configuration and moduli spaces

In this section we would like to discuss the gauge symmetries under which the superpotential $\mathcal{W}$ is left invariant and consider the resulting reduced configuration space and the associated reduced subspace of the space $\mathcal{C}_{\mathrm{hol}}$ of critical points of $\mathcal{W}$. Clearly, if we parametrize the possible deformations of the world-volume field-strength $\mathcal{F}$ with a one-form a as we have explained in section 4 (and more extensively in appendix A), then any transformation generated by an exact $a = d\lambda$, with λ some function on Σ, is a gauge symmetry of $\mathcal{W}$. Following the previous discussions, the above gauge symmetry generated by λ can be identified with the tangent vector field $X_\lambda \in \Gamma(T_\Sigma)$ such that $d\lambda = (P[g] + \mathcal{F}) \cdot X_\lambda$, that can in turn be seen as a vector tangent to $\mathcal{C}$ at $(\Sigma, \mathcal{F})$. Call $\mathbf{g}$ the subbundle of $T_\mathcal{C}$ spanned by such vectors X_λ. Now, the holomorphicity of $\mathcal{W}$ with respect to the almost complex

structure $\mathbb{J}$ automatically implies that $\mathcal{W}$ is not only left invariant by the general X_λ defined above, but also under its image $\mathbb{J}X_\lambda$ under $\mathbb{J}$, that we can consider as its imaginary extension. This means that $\mathcal{W}$ is invariant under the action of the general section of the subbundle $\mathbf{g}^\mathbb{C}$ generated by the vectors of the form X_λ and $\mathbb{J}X_\lambda$. Indeed, the holomorphicity of $\mathcal{W}$ implies that, for any $Y \in \Gamma(T_\mathcal{C})$,

$$(1 + i\mathbb{J})Y(\mathcal{W}) \equiv 0 \,, \tag{8.1}$$

and then

$$X_\lambda(\mathcal{W}) \equiv 0 \Rightarrow \mathbb{J}X_\lambda(\mathcal{W}) \equiv 0 \ . \tag{8.2}$$

It can be clarifying to see how this "complexification" of the natural $u(1)$ gauge symmetry of the internal generalized cycle $(\Sigma, \mathcal{F})$ reduces to standard ones when we restrict to the well studied subcases of A and B branes on Calabi-Yau 3-folds. In the Calabi-Yau case, J_1 is equal to J_2 and is the proper (integrable) complex structure of the Calabi-Yau. Consider first B-branes. These wrap holomorphic cycles with holomorphic connections A on them (such that $\mathcal{F} = dA$). In this case, $\mathbb{J}X_\lambda$ generate the transformation $\delta A = i(\partial\lambda - \bar\partial\lambda)$ which is properly identified as an imaginary transformation of the complexified gauge algebra $u(1)^\mathbb{C} = \mathbb{C}^*$. Secondly, consider a Lagrangian A-branes Σ, with U(1) flat connection A (such that $\mathcal{F} = dA = 0$). In this case $\mathbb{J}X_\lambda$ is associated to a normal vector field of the form $J_1 P[g]^{-1} d\lambda$, which corresponds exactly to the general normal vector field generating Hamiltonian deformations of the Lagrangian A-brane, that must be indeed considered as gauge symmetries relating equivalent Lagrangians. We then see how our formalism include these specific subcases and provide their natural extension to less trivial $\mathcal{N} = 1$ (and $\mathcal{N} = 2$) flux compactifications.

Note that in the above example with A and B branes, we have really restricted to the space $\mathcal{C}_{\text{hol}}$, while analysis presented above is valid for the whole $\mathcal{M}$. This has been possible due to the property that $T_{\mathcal{C}_{\text{hol}}}$ is clearly stable under the action of $\mathbb{J}$ and then $\mathcal{C}_{\text{hol}}$ is preserved under the action of $\mathbf{g}^\mathbb{C}$.[11] The subspace $\mathcal{C}_{\text{hol}}$ is also special because, as we have discussed in section 7, the almost complex structure $\mathbb{J}$ restricted to it can be defined using only the integrable generalized complex structure on M, without any need to involve the $SU(3) \times SU(3)$ structure. This property implies that the generalized complex structure $\mathcal{J}_2$ on M naturally induces an almost complex structure on $\mathcal{C}_{\text{hol}}$. Note that, since only the integrable generalized complex structure is involved in this definition, all the discussion can be adapted to the case in which we consider topological branes of the underlying topological model [11, 5].

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[11]The subspace $\mathcal{C}_{\text{hol}}$ can be defined by the condition $d\mathcal{W}|_{\mathcal{C}_{\text{hol}}} = 0$ and then a vector $X \in T_\mathcal{C}|_{\mathcal{C}_{\text{hol}}}$ tangent to $\mathcal{C}_{\text{hol}}$ can be defined by the condition

$$d[X(\mathcal{W})]|_{\mathcal{C}_{\text{hol}}} = 0 \,, \tag{8.3}$$

where, in each point $(\Sigma, \mathcal{F}) \in \mathcal{C}_{\text{hol}}$, we consider X as a field obtained extending a vector $X \in T_\mathcal{C}|_{(\Sigma, \mathcal{F})}$ to a neighborhood of $(\Sigma, \mathcal{F})$ (of course the condition (8.3) does not depend on the choice of the extension). From the holomorphicity of the superpotential one can thus conclude that if X is tangent to $\mathcal{C}_{\text{hol}}$ then also $\mathbb{J}X$ is tangent to $\mathcal{C}_{\text{hol}}$. This means that $T_{\mathcal{C}_{\text{hol}}}$ is stable under the action of $\mathbb{J}$. Then, since $\mathcal{C}_{\text{hol}}$ is stable under the action of $\mathbf{g}$, it is also stable under the action of $\mathbf{g}^\mathbb{C}$.

The natural question is if such an almost complex structure on $\mathcal{C}_{\text{hol}}$ is actually integrable. Unfortunately, already in the case of standard Calabi-Yau compactifications the answer is not well understood in general. For example, one can introduce an almost complex structure of the space of Lagrangian submanifolds (with flat U(1) connection) using the symplectic structure of the Calabi-Yau. The resulting almost complex structure mixes embedding and gauge "coordinates", and its integrability issue is still not clear[12] (see for example [22]). Since our analysis includes this special subcase, we do not try to give an answer to the problem in the present paper. It would be interesting to understand better this issue from the generalized geometry point of view, that appears to be the natural complex-symplectic unifying language to better approach it.

Finally, observe that one can use $\mathbb{J}$ to naturally induce an almost complex structure on $\mathcal{C}^{\text{red}} = \mathcal{C}/\mathcal{G}^{\mathbb{C}}$, where $\mathcal{G}^{\mathbb{C}}$ is the group of finite gauge transformations generated by $\mathbf{g}^{\mathbb{C}}$. Furthermore, since the superpotential $\mathcal{W}$ is left invariant by the action of $\mathcal{G}^{\mathbb{C}}$, we can also introduce an almost complex structure on the quotient space $\mathcal{M} = \mathcal{C}_{\text{hol}}/\mathcal{G}^{\mathbb{C}}$. As will be clear from the discussion of the following section, $\mathcal{M}$ can be identified as the moduli space of the supersymmetric configurations of a space-time filling D-brane. Furthermore, it is known that in the case of Lagrangian branes on ordinary Calabi-Yau 3-folds, the above almost complex structure on $\mathcal{C}_{\text{hol}}$ descends to an *integrable* complex structure on the corresponding $\mathcal{M}$ (i.e. on the moduli space of special Lagrangian branes). Thus, it seems plausible to hope that the above almost complex structure on $\mathcal{M}$ can be in fact integrable also in the most general case. We postpone the investigation of this interesting problem to future investigations.

9. D-flatness and moment map

In this section we will consider more closely the supersymmetry D-flatness condition written in (2.16). As we already discussed in section 3 this condition can be indeed seen as coming from the vanishing of a D-term associated to the effective four-dimensional theory. As we will now see, the D-flatness condition provide a gauge fixing slice for the action of the imaginary extension of the gauge group, and then select a particular hypersurface $\mathcal{C}_0$ in $\mathcal{C}$. The action of $\mathcal{G}$ foliates $\mathcal{C}_0$ in gauge orbits and the base of such a foliation can be identified as the reduced moduli space $\mathcal{M}$.

The argument is based on the possibility to see the D-flatness condition as the vanishing of a moment map associated to the gauge transformation discussed in the previous section, defined with respect to a properly introduced symplectic form. The approach is completely analogous to the one used in the study of branes in Calabi-Yau spaces (see e.g. Chapter 38 of [23] for a review), even if it differs from it in some details. Let us start by introducing the following formal symplectic structure on $\mathcal{C}$. Looking at the vectors $X, Y \in T_{\mathcal{C}}|_{(\Sigma,\mathcal{F})}$ as sections of $T_M|_\Sigma$ by the usual identification, we introduce the following symplectic form

$$\Xi(X,Y)|_{(\Sigma,\mathcal{F})} = \int_\Sigma X^m Y^n P[e^{2A-\Phi}(\hat{\gamma}_{mn} + \mathcal{F}_{mn})\text{Im}\hat{\Psi}_1] \wedge e^{\mathcal{F}} , \qquad (9.1)$$

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[12]I thank R.P. Thomas for correspondence on this point.

where with $\mathcal{F}_{mn}$ we mean the natural extension with zero orthogonal components of the world-volume field-strength $\mathcal{F}$ to the complete $T_M|_\Sigma$, and we recall that the six dimensional gamma matrices $\hat{\gamma}_m$ act on a form ω as follows

$$\hat{\gamma}_m \omega = (\imath_m + g_{mn} dy^n \wedge) \omega . \tag{9.2}$$

It is easy to see that using the alternative coordinatization for $T_\mathcal{C}$ given by $(X_\perp, a)$ and $(Y_\perp, b)$ associated to X and Y respectively by (7.6), the above symplectic form takes the form

$$\Xi[(X_\perp, a), (Y_\perp, b)]|_{(\Sigma, \mathcal{F})} = \int_\Sigma \Big\{ a \wedge b \wedge P[e^{2A-\Phi} \mathrm{Im}\hat{\Psi}_1] + P[e^{2A-\Phi} \imath_{X_\perp} \imath_{Y_\perp} \mathrm{Im}\hat{\Psi}_1] + \tag{9.3}$$
$$+ a \wedge P[e^{2A-\Phi} \imath_{Y_\perp} \mathrm{Im}\hat{\Psi}_1] - b \wedge P[e^{2A-\Phi} \imath_{X_\perp} \mathrm{Im}\hat{\Psi}_1] \Big\} \wedge e^{\mathcal{F}} .$$

Note that if we restrict to the case of D-branes on Calabi-Yau manifolds, the above symplectic structure coincides with the Kähler forms constructed for A and B branes. As in that case, we will not worry whether Ξ is closed or not, since it will not really be relevant for what follows (for discussions on this point see [22]). Note also that, in our general case, Ξ cannot be seen as the Kähler form Θ that can be constructed from the metric G defined in (3.6) and the complex structure $\mathbb{J}$ (i.e. $\Theta(X, Y) = G(X, \mathbb{J}Y)$). However Ξ and Θ are related in the following way

$$\Theta(X, Y)|_{(\Sigma, \mathcal{F})} = \frac{1}{2} \big[\Xi(X, Y) + \Xi(\mathbb{J}X, \mathbb{J}Y) \big]|_{(\Sigma, \mathcal{F})} +$$
$$- \frac{1}{2} \int_\Sigma \{ \mathcal{F}(X, Y) + \mathcal{F}(JX, JY) \} P[e^{2A-\Phi} \mathrm{Im}\hat{\Psi}_1] \wedge e^{\mathcal{F}} . \tag{9.4}$$

We can now introduce the moment map $m : \mathcal{C} \to \Gamma(\Lambda^{\mathrm{top}} T_\Sigma^*)$ as follows

$$m(\Sigma, \mathcal{F}) = P[e^{2A-\Phi} \mathrm{Im}\hat{\Psi}_1] \wedge e^{\mathcal{F}}|_{\mathrm{top}} . \tag{9.5}$$

The moment map m associates any world-volume function λ generating a gauge transformation to the corresponding Hamiltonian function (with respect to the symplectic form Ξ) given by the pairing

$$\langle m(\Sigma, \mathcal{F}), \lambda \rangle = \int_\Sigma \lambda P[e^{2A-\Phi} \mathrm{Im}\hat{\Psi}_1] \wedge e^{\mathcal{F}} . \tag{9.6}$$

To prove it, it is sufficient to verify that, for any vector $Y \in T_\mathcal{C}|_{(\Sigma, \mathcal{F})}$, we have

$$d\langle m(\Sigma, \mathcal{F}), \lambda \rangle(Y) = \Xi(X_\lambda, Y) , \tag{9.7}$$

where $X_\lambda \in \Gamma(T_\Sigma)$ is the vector generating the gauge transformation and is defined by the relation $d\lambda = (P[g] + \mathcal{F}) \cdot X_\lambda$ (see section 8). We can then conclude that the D-flatness condition in (2.16) can be seen as the restriction to the subspace of $\mathcal{C}$ given by $m^{-1}(0)$. It is clear that any (real) gauge transformation λ preserves the constraint $m(\Sigma, \mathcal{F}) = 0$, since for any h we have that

$$X_\lambda(\langle m(\Sigma, \mathcal{F}), h \rangle) = \Xi(X_h, X_\lambda) = \int_\Sigma dh \wedge d\lambda \wedge P[e^{2A-\Phi} \mathrm{Im}\hat{\Psi}_1] \wedge e^{\mathcal{F}} \equiv 0 , \tag{9.8}$$

where we have used the d_H-closedness of $e^{2A-\Phi}\mathrm{Im}\hat{\Psi}_1$. On the other hand, it is easy to see that $m^{-1}(0)$ does provide a gauge fixing section for the imaginary gauge transformations. To show this, consider the general imaginary gauge transformation generated by a vector of the form $\mathbb{J}X_\lambda$, with λ a general world-volume function as before. Then one can readily realize that if $(\Sigma, \mathcal{F}) \in m^{-1}(0)$ then $(\mathbb{J}X_\lambda)(\langle m(\Sigma, \mathcal{F}), h\rangle)$ cannot vanish for any h. To see it, it is enough to take $h = \lambda$: using the relation (9.4) to relate Ξ to the Kähler form Θ, we have that

$$(\mathbb{J}X_\lambda)(\langle m(\Sigma, \mathcal{F}), \lambda\rangle)|_{m^{-1}(0)} = \Xi(X_\lambda, \mathbb{J}X_\lambda)|_{m^{-1}(0)} = \Theta(X_\lambda, \mathbb{J}X_\lambda)|_{m^{-1}(0)}$$
$$= -G(X_\lambda, X_\lambda)|_{m^{-1}(0)}, \tag{9.9}$$

which generally never vanishes.

Let us note that in the definition of the symplectic structure (9.3) we have used in an essential way the background metric to again identify $\mathcal{N}_{(\Sigma, \mathcal{F})}$ with $T_\Sigma^\perp \oplus T_\Sigma^*$. This is analogous to what happens in the definition of the almost complex structure $\mathbb{J}$ defined in section 7. However, analogously to what happens that case, it is easy to see that if we restrict to $m^{-1}(0)$ the symplectic structure (9.3) is canonically defined on sections of $\mathcal{N}_{(\Sigma, \mathcal{F})}$, in the sense that does not depend on the choice of the subbundle of $T_M \oplus T_M^*|_\Sigma$ that should represent $\mathcal{N}_{(\Sigma, \mathcal{F})}$ in a particular 'gauge'.

Then, to summarize, the D-flatness condition in (2.16) can be written in the form $m(\Sigma, \mathcal{F}) = 0$ and clearly provide a global section for the imaginary gauge transformations described in section 8. The resulting constrained space $m^{-1}(0)$ is closed under real gauge orbits generated by $\mathcal{G}$ and the quotient space $m^{-1}(0)/\mathcal{G}$ provide a characterization of the reduced configuration space $\mathcal{C}^{\mathrm{red}} = \mathcal{C}/\mathcal{G}^{\mathbb{C}}$. Furthermore, the same conclusions can be reached if we restrict to the space $\mathcal{C}_{\mathrm{hol}}$ of generalized complex submanifolds, and then we can make the identifications $\mathcal{M} = \mathcal{C}_{\mathrm{hol}}/\mathcal{G}^{\mathbb{C}} = [\mathcal{C}_{\mathrm{hol}} \cap m^{-1}(0)]/\mathcal{G}$. Whether in each orbit of $\mathcal{G}^{\mathbb{C}}$ inside $\mathcal{C}_{\mathrm{hol}}$ there exists or not a $\mathcal{G}$ orbit satisfying the D-flatness condition (and thus minimizing the four-dimensional energy density) can be seen as a generalization of the standard formulation of the stability problem that would be interesting to understand better in the present context. Finally, we have stressed that (9.3) can be only formally considered a symplectic form, since it is in general non-closed. However, from the knowledge of what happens in the standard Calabi-Yau case, it is possible to expect that the closedness can be recovered by restricting to $\mathcal{M}$. As the issue of the integrability of the almost complex structure $\mathbb{J}$, the problem to understand in what sense we can consider the symplectic structure (9.3) as actually closed requires further investigations.

10. Examples and applications for D-branes in SU(3)-structure vacua

In this section we will consider some basic examples where we can apply explicitly the analysis presented in the previous sections. In particular, we will restrict a little the general setting by focusing on supersymmetric backgrounds with internal SU(3)-structure, which are the closest to ordinary flux-less compactifications on Calabi-Yau three-folds. Let

us review some of their properties [28, 5, 56]. The SU(3)-structure vacua are characterized by the property that the two internal Weyl spinors $\eta_+^{(1)}$ and $\eta_+^{(2)}$ are actually proportional. It means that we can write them as $\eta^{(1)} = a\eta_+$ and $\eta^{(2)} = b\eta_+$, in terms of a single internal spinor η_+, such that $\eta_+^\dagger \eta_+ = 1$. As we have recalled in section 2, since we are considering D-calibrated backgrounds, we must furthermore impose that $|a| = |b|$. Thus we pose

$$a = e^{i\varphi_1}|a|, \qquad b = e^{i\varphi_2}|a| . \tag{10.1}$$

From η_+ one can construct an almost complex structure J (with respect to which the internal metric is hermitian) and a $(3,0)$ form Ω in the following way

$$J^m{}_n = -\frac{i}{|a|^2}\eta_+^\dagger \hat{\gamma}^m{}_n\eta\,, \qquad \Omega_{mnp} = -\frac{i}{a^2}\eta_-^\dagger \hat{\gamma}_{mnp}\eta_+ . \tag{10.2}$$

J and Ω have all the algebraic properties of the complex structure and the holomorphic three form on a standard Calabi-Yau three-fold (see [57] for a review). In this case the two normalized pure spinors (2.13) become

$$\hat{\Psi}^+ = -ie^{i(\varphi_1 - \varphi_2)}e^{-iJ}\,, \qquad \hat{\Psi}^- = -e^{i(\varphi_1 + \varphi_2)}\Omega . \tag{10.3}$$

Here and in the following we use J to indicate also the Kähler form associated to the almost complex structure (the actual meaning being clear from the context). In the particular $\mathcal{N} = 2$ subcase in which the internal space is a standard Calabi-Yau, the expressions (10.3) for the pure spinors are still valid, but with constant arbitrary overall phases.

From (10.3) it follows that SU(3)-structure backgrounds are somehow special in the whole family of SU(3) × SU(3)-structure backgrounds: in Type IIB the internal space is actually complex with $c_1(M) = 0$ while in Type IIA the internal manifold is symplectic. In the following examples we will re-obtain these and other needed properties of the SU(3)-structure backgrounds [28, 5] directly from the supersymmetry conditions (2.8). In this way, we will have a further case-by-case check of the deep relation between the $\mathcal{N} = 1$ backgrounds we are considering and the supersymmetric D-branes they admit. Namely, we will focus on D3, D5, D6 and D7-branes, with particular attention to this last case. Supersymmetric D4-branes are not allowed in Type IIA SU(3)-structure backgrounds [8]. D8- and D9-branes can be analyzed along the same lines of the cases explicitly discussed below. Let us make only a comment on the D9-brane case. In this case, we can write the superpotential (4.9) by thinking as we had one more dimension. Furthermore the non-abelian generalization is straightforward in this case and simply replaces $\mathcal{F} \wedge \mathcal{F}$ with the non-abelian analogous $\text{Tr}\mathcal{F} \wedge \mathcal{F}$. Then, if we consider the case of an internal flux-less Calabi-Yau and $\mathcal{F} = dA + A \wedge A$, the resulting superpotential becomes up to a constant

$$\mathcal{W} = \frac{1}{4}\int_{\text{CY}_3} \Omega \wedge \text{Tr}\left(A \wedge \bar{\partial}A + \frac{2}{3}A \wedge A \wedge A\right), \tag{10.4}$$

thus reproducing the Witten's Chern-Simons theory describing B-branes filling a Calabi-Yau three-fold [3].

10.1 D3-branes

If we consider the simplest case of D3-branes in a SU(3)-structure Type IIB background, the space of possible configurations corresponds to the internal space itself, and then has naturally a complex structure. However, in general the configuration space is not Kähler without imposing some further condition.

The superpotential (4.9) for D3-branes vanishes identically and thus the F-flatness condition is always satisfied. On the other hand, the D-flatness condition is simply given by

$$\cos(\varphi_1 - \varphi_2)|_{y_0} = 0 \quad \Leftrightarrow \quad (a \pm ib)|_{y_0} = 0, \tag{10.5}$$

where y_0 is the point of the internal manifolds where the D3-brane is located and the actual sign on the right-hand side of (10.5) depends on the orientation of the D3-brane. Note that from the first background condition in (2.8) one obtains that $d[e^{2A-\Phi} \cos(\varphi_1 - \varphi_2)] = 0$ and thus, if the condition (10.5) is satisfied in a point, it is satisfied everywhere. This is consistent with the fact that in the case of a single D3-brane we do not have any charged matter field under the gauge group and then we cannot have any non-trivial D-term around a supersymmetric configuration.

The condition $a = \pm ib$ characterizes the so-called type B backgrounds, first considered in [58–60], which constitute the supersymmetric subsector of the class of supergravity solutions discussed in [61] (see also the recent review [4]). In this case, the second condition in (2.8) translates into the following two conditions

$$d\tilde{\Omega} = 0, \qquad \tilde{\Omega} \wedge H = 0 . \tag{10.6}$$

where, to stress the analogy with the standard Calabi-Yau case, we have introduced the holomorphic $(3,0)$-form $\tilde{\Omega}$ defined as

$$\tilde{\Omega} = -e^{3A-\Phi}e^{2i\varphi_1}\Omega . \tag{10.7}$$

The first condition in (10.6) tells us that all these backgrounds (like all the other SU(3)-structure Type IIB vacua) are actually complex. The second condition simply means that H has only $(2,1)$ and $(1,2)$ components. Looking now at the real part of the first supersymmetry condition in (2.8), we obtain the conditions

$$d(e^{2A-\Phi}J) = 0, \qquad H \wedge J = 0 . \tag{10.8}$$

The first condition implies that the internal space is a warped Kähler space, with warp-factor $e^{-2A+\Phi}$, so that the closed Kähler form is given by $J^{(K)} = e^{2A-\Phi}J$. The second condition in (10.8), together with the second condition in (10.6), are part of the more general requirement that, in these type B solutions, the complex three form $G_{(3)} = F_{(3)} - \tau H$ (where $\tau = C_{(0)} + ie^{-\Phi}$) must be $(2,1)$ and primitive. To obtain the cases in which the internal space is actually a warped Calabi-Yau [58, 59], one must impose that the dilaton is constant (actually τ must be constant). This condition can be easily obtained by requiring that in the Calabi-Yau case $J^{(K)} \wedge J^{(K)} \wedge J^{(K)}$ must be proportional, up to a constant factor, to $i\tilde{\Omega} \wedge \bar{\tilde{\Omega}}$.

From this short review of some of the main properties of the type B vacua, we reach the conclusion that, if the moduli space of supersymmetric D3-branes must coincide with the internal manifold itself, then it is automatically a Kälher manifold that, if we furthermore require a constant dilaton, is also Calabi-Yau.

10.2 D5-branes

In the case of D5-branes wrapping an internal two-cycle Σ the superpotential (4.9) takes the form of the Witten's superpotential [17]

$$\mathcal{W} = \mathcal{W}_0 + \frac{1}{2} \int_{\mathcal{B}} P[\tilde{\Omega}] \,, \tag{10.9}$$

where again we used the holomorphic $(3,0)$-form $\tilde{\Omega} \equiv e^{3A-\Phi}\hat{\Psi}^-$. By considering a complex coordinatization z^i $(i = 1, 2, 3)$ of the internal space, we can specify the embedding using complex fields $\phi^i(\sigma)$ (where σ^α are world-volume coordinates). The superpotential is clearly holomorphic with respect to these complex fields. However, if we want to consider the superpotential as a functional on the space of diffeomorphism equivalent cycles, the background complex structure does not naturally induce a complex structure for it, and we need some additional structure. Indeed, the almost complex structure $\mathbb{J}$ introduced in section 7 uses two additional ingredients: the background metric that allows to identify explicitly the normal bundle of the two-cycle with its orthogonal bundle, and the world-volume gauge field, which is in general mixed with the embedding coordinates under the action of the almost complex structure.

Turning to the D-flatness condition, it takes the form

$$\mathcal{F} = -\mathrm{tg}(\varphi_1 - \varphi_2) P_\Sigma[J] \,. \tag{10.10}$$

Then, if one wants to admit supersymmetric D5-branes with zero $\mathcal{F}$, it is natural to impose everywhere the condition $\varphi_1 - \varphi_2 = 0$ or π, or equivalently $a = \pm b$ (more in general, it would be sufficient to impose such a condition only where the brane is located). This condition defines the so called type C backgrounds (see e.g. [4] for more on them), of which the Maldacena-Nuñez solution [62] provides the most known explicit example. A different and somehow special case is obtained by considering a type B background, that is $\varphi_1 - \varphi_2 = \pm\frac{\pi}{2}$. The D5-brane can then be supersymmetric only if it wraps a collapsed cycle (so that $P_\Sigma[J] = 0$) with a non-vanishing $\mathcal{F}$ field on it, in such a way to have a non-vanishing tension. The resulting configurations are fractional D3-branes, that are well known supersymmetric configurations giving rise to corresponding backgrounds with fluxes.

Note that, in the case of type C solutions (fixing for example $\varphi_1 - \varphi_2 = 0$), from the real part of the first supersymmetry condition in (2.8) one can directly obtain the conditions $d(2A - \Phi) = 0$ and $H = 0$. Thus, it is not difficult to see that the symplectic form (9.3) takes the form

$$\Xi(X, Y) = -e^{2A-\Phi} \int_\Sigma \left\{ a \wedge b + \frac{1}{2} P[\imath_{Y_\perp} \imath_{X_\perp}(J \wedge J)] \right\} \tag{10.11}$$

where we have moved $e^{2A-\Phi}$ out of the integral since it is constant. If moreover we restrict to the case of holomorphic two-cycles, this symplectic form becomes

$$\Xi(X,Y) = -e^{2A-\Phi} \int_\Sigma \left\{ a \wedge b + J(X_\perp, Y_\perp) P[J] \right\} . \tag{10.12}$$

One can immediately check that the moment map for the ordinary gauge transformations $a = d\lambda$ is given by $m(\Sigma, \mathcal{F}) = -e^{2A-\Phi}\mathcal{F}$, consistent with our general discussion. Imposing that it must vanish is of course equivalent to the D-flatness condition (10.10). Note that the restricted Ξ given in (10.12) can be directly related to the Kähler two-from Θ (see section 9) in the following way

$$\Xi(X,Y) = \Theta(X,Y) - e^{2A-\Phi} \int_\Sigma \mathcal{F}(X,Y)\mathcal{F} . \tag{10.13}$$

Note that, also comparing with the general formula (9.4), in this case we have clearly that $\Xi(X,Y)$ is of the type $(1,1)$. Thus, it can be seen as a deformation of the Kähler form Θ due to the presence of nontrivial world-volume $\mathcal{F}$.

10.3 D6-branes

Let us now consider the case of a D6-brane wrapping an internal three-cycle in a Type IIA SU(3)-structure background. Note first of all that, from the second supersymmetry condition in (2.8), we immediately obtain that $3A - \Phi + i(\varphi_1 - \varphi_2)$ must be constant, H must vanish, and $dJ = 0$. This explicitly checks the known property that the internal space must be symplectic (but in general not complex). Thus, we can write the superpotential for D6-branes in the following explicit form

$$\mathcal{W} = \mathcal{W}_0 - \frac{1}{2} e^{3A-\Phi+i(\varphi_1-\varphi_2)} \int_\mathcal{B} \left\{ P[J] \wedge \tilde{\mathcal{F}} + \frac{i}{2} P[J \wedge J] - \frac{i}{2} \tilde{\mathcal{F}} \wedge \tilde{\mathcal{F}} \right\} \tag{10.14}$$

One can easily check that in this case a generalized complex three-cycle corresponds to a Lagrangian submanifold with vanishing field-strength, i.e.

$$P_\Sigma[J] = 0 , \qquad \mathcal{F} = 0 . \tag{10.15}$$

Note that (10.14) is completely identical in form to the holomorphic functional presented for example in [22], that can be written in the form of the standard Chern-Simons action that was proved in [3] to describe Lagrangian A-branes.

Let us now consider the D-flatness condition, which reads

$$P_\Sigma[\text{Im}\tilde{\Omega}] = 0 . \tag{10.16}$$

where again we have posed $\tilde{\Omega} = e^{2A-\Phi}e^{i(\varphi_1+\varphi_2)}\Omega$, which obeys the condition $d(\text{Im}\tilde{\Omega}) = 0$. Note that in this case we do not have any immediate constraint to be imposed on the background in order for it to admit a supersymmetric D6-brane. The only obvious necessary condition is the following topological condition that must be imposed on the brane

$$\int P_\Sigma[\text{Im}\tilde{\Omega}] = 0 . \tag{10.17}$$

The formal symplectic form (9.3), when evaluated in a general point of the configuration space $\mathcal{C}$, is explicitly given by

$$\Xi(X,Y) = \int_\Sigma \left\{ a \wedge P[\imath_{Y_\perp} \mathrm{Im}\tilde\Omega] - b \wedge P[\imath_{X_\perp} \mathrm{Im}\tilde\Omega] + P[\imath_{X_\perp} \imath_{Y_\perp} \mathrm{Im}\tilde\Omega] \wedge \mathcal{F} \right\} . \quad (10.18)$$

If we now restrict to the superpotential critical subspace $\mathcal{C}_{\mathrm{hol}}$ of Lagrangian cycles, the symplectic form reduces to

$$\Xi(X,Y) = \int_\Sigma \left\{ a \wedge P[\imath_{Y_\perp} \mathrm{Im}\tilde\Omega] - b \wedge P[\imath_{X_\perp} \mathrm{Im}\tilde\Omega] \right\} . \quad (10.19)$$

This is identical in form to the symplectic form introduced in [22] for Lagrangian branes with flat U(1) connection on ordinary Calabi-Yau three-folds.

10.4 D7-branes

The final case that we analyze explicitly is that of a D7-brane in a Type II SU(3)-structure background. In order to be more concrete, let us focus on the case in which the background is of the type B described in the discussion about D3-branes. As we have already said, these backgrounds can be thought of as generated by D3 and/or fractional D3 and/or D7-branes [60] and their internal manifold has a warped Kähler metric with Kähler form $J^{(\mathrm{K})} = e^{2A-\Phi}J$ and a global holomorphic $(3,0)$-form $\tilde\Omega$ defined in (10.7). This will allow us to discuss some interesting additional issues related to the moduli space of the D7-brane.

In type B backgrounds, the superpotential for the D7-brane is given by

$$\mathcal{W}(\Sigma, \mathcal{F}) = \mathcal{W}_0 + \frac{1}{2} \int_\mathcal{B} P[\tilde\Omega] \wedge \tilde{\mathcal{F}} . \quad (10.20)$$

We already know that the extrema of this superpotential are given by D-branes wrapping generalized complex submanifolds $(\Sigma, \mathcal{F})$, with Σ holomorphically embedded and $\mathcal{F}$ of kind $(1,1)$. Let us see this directly from the superpotential (10.20). The variation with respect of the world-volume gauge field gives the condition

$$P_\Sigma[\tilde\Omega] = 0 . \quad (10.21)$$

This condition requires the cycle Σ to be holomorphically embedded. The additional F-flatness condition that $\mathcal{F}$ must be of kind $(1,1)$ can be derived by varying the embedding coordinates along the vector field $X \in \Gamma(T_M|_\Sigma)$ and transforming the world-volume field strength accordingly to the rule $\delta\mathcal{F} = P_\Sigma[\imath_X H]$. The resulting derivative of the superpotential is given by

$$P_\Sigma[\imath_X \tilde\Omega] \wedge \mathcal{F} , \quad (10.22)$$

which clearly vanishes only if $\mathcal{F}_{(0,2)} = 0$. Note that, as we have already discussed in general in section 4, once we take into account the other condition (10.21), the condition (10.22) is well defined also thinking to X as a section of the canonical normal bundle $\mathcal{N}_\Sigma = T_M|_\Sigma/T_\Sigma$.

Now, in principle the superpotential (10.20) takes into account all the possible internal fluctuation modes of the D7-brane. One could then 'integrate out' the heavy massive modes,

to obtain an effective superpotential for the light ones, as described in [3]. More directly, the superpotential (10.20) allows to immediatly discuss a mechanism of flux-generated lifting of the possible moduli fields corresponding to the infinitesimal deformations of a holomorphic cycle. Such an effect was discussed in [32] using a different procedure in the less general case where the internal manifold is a warped Calabi-Yau. The following discussion generalizes it and clarifies its origin.

Let us first recall that the possible infinitesimal deformations of a holomorphic cycle Σ (of arbitrary dimension) are given by the space of global sections $H^0(\Sigma, \mathcal{N}_\Sigma^{\text{hol}})$ of the holomorphic normal bundle $\mathcal{N}_\Sigma^{\text{hol}} = T_M^{1,0}|_\Sigma / T_\Sigma^{1,0}$. In our case, Σ is a divisor. By the triviality of the canonical bundle of M and the adjunction formula, one obtains the standard result that $H^0(\Sigma, \mathcal{N}_\Sigma^{\text{hol}}) = H^{2,0}(\Sigma)$. Thus, there are $h^{2,0}(\Sigma) = \dim H^{2,0}(\Sigma)$ possible moduli deformations parametrized by complex coordinates t^i, $i = 1, \ldots, h^{2,0}(\Sigma)$.

The first order derivative by t^i of the superpotential (10.20) is given by

$$\partial_i \mathcal{W} = \frac{1}{2} \int_\Sigma P_\Sigma[\imath_{X_i}\tilde{\Omega}] \wedge \mathcal{F}, \tag{10.23}$$

where X_i is the holomorphic section of $\mathcal{N}_\Sigma^{\text{hol}}$ generating the shift in t^i. Obviously (10.23) vanishes in a point t_0 where $\mathcal{F}$ is $(1,1)$ and in general one obtains a set of $h^{2,0}(\Sigma)$ possible moduli lifting conditions

$$a_i(t) \equiv \int_\Sigma P_\Sigma[\imath_{X_i}\tilde{\Omega}] \wedge \mathcal{F} = 0, \tag{10.24}$$

that can in principle lift all the possible $h^{2,0}(\Sigma)$ moduli fields t^i. In [32], the set of $h^{2,0}(\Sigma)$ conditions $a_i(t) = 0$ were found by a rather different way in the warped Calabi-Yau subcase, conjecturing that the a_i's could be identified as the first derivatives of a superpotential. Equation (10.23) gives a direct confirmation and generalization of that proposal.

Possible holomorphic mass terms can be now in principle computed by taking a further derivative of the superpotential. Let us first of all recall that, already in the flux-less Calabi-Yau case, when $T_M^{1,0}|_\Sigma$ does not holomorphically split into $T_\Sigma^{1,0} \oplus \mathcal{N}_\Sigma^{\text{hol}}$ some of the $h^{2,0}(\Sigma)$ infinitesimal embedding deformations may be in fact massive, due to possible obstructions coming from the holomorphic line bundle on the brane [64]. The superpotential (10.20) directly exhibits the possible presence of this kind of obstructions, even in the more general case of backgrounds with fluxes we are considering. Indeed, the variation of $P_\Sigma[\imath_{X_i}\tilde{\Omega}]$ in (10.23) may produce in general a $(1,1)$ form that, combined with a non-trivial $(1,1)$ world-volume field-strength $\mathcal{F}$, can give non-vanishing mass terms for the t^i's.[13]

In order to focus on mass terms that are a peculiar effect of the background fluxes, let us now assume the holomorphic splitting of $T_M^{1,0}|_\Sigma$ into $T_\Sigma^{1,0} \oplus \mathcal{N}_\Sigma^{\text{hol}}$. In this case, when the internal manifold is a standard flux-less Calabi-Yau, the t^i's are massless, even if there can be possible higher order obstructions (like the standard ones that lies in $H^1(\Sigma, \mathcal{N}_\Sigma^{\text{hol}})$ [63]) that should be described by non-trivial higher order terms in the superpotential (10.20) (see for example the related discussions in [10, 18, 19]). However, from the superpotential (4.9)

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[13]This observation is due to F. Denef, who I thank for discussions on this point.

one can easily realize that, in presence of a non-trivial background H-flux, the t^i's can in general cease to be massless. To see this, it is enough to take the second derivative of the superpotential around a point t_0 corresponding to a generalized holomorphic cycle $(\Sigma_0, \mathcal{F}_0)$, obtaining the following holomorphic mass matrix

$$m_{ij}(t_0) \equiv (\partial_i \partial_j \mathcal{W})(t_0) = \frac{1}{2} \int_{\Sigma_0} P_{\Sigma_0}[\imath_{X_i} \tilde{\Omega} \wedge \imath_{X_j} H] \, . \tag{10.25}$$

The formula (10.25) explicitly shows how a nontrivial H-flux can induce holomorphic mass terms (that would otherwise vanish) for the possible embedding moduli.

If the F-flatness conditions are satisfied, in order to obtain a full supersymmetric configuration we have still to impose the D-flatness condition, that in this case reads

$$P_\Sigma[J] \wedge \mathcal{F} = 0 \, . \tag{10.26}$$

Such a condition is a generalization of what is known as a Hermitian-Yang-Mills condition in standard Yang-Mills theories. We can then easily adapt the standard argument for an abelian Yang-Mills theory (see for example [65]) to prove that in each orbit of different $\mathcal{F}$'s generated by the imaginary extension of the (abelian) gauge group there is a particular $\mathcal{F}$ satisfying the D-term condition (10.26) if and only if the condition

$$\int_\Sigma P[e^{2A - \Phi} J] \wedge \mathcal{F} = 0 \, , \tag{10.27}$$

is satisfied. Indeed the imaginary gauge transformation acts as $\delta \mathcal{F} = i \partial \bar{\partial} \lambda$, where λ is any real function on Σ and ∂ and $\bar{\partial}$ are the standard Dolbeault differential operators on Σ. It immediately follows that the condition (10.27) is necessary and sufficient for the existence of a λ such that the transformed $\mathcal{F}$ satisfies (10.26). Note also that the condition (10.27) is actually topological (in the sense that it is left invariant by any continuous deformation of Σ and $\mathcal{F}$), due to the primitivity condition $J \wedge H = 0$.

Also, from the general discussion of section 9, we know that the D-flatness condition (10.27) can be obtained as the vanishing moment map condition associated to the symplectic form (9.3). Restricting to supersymmetric configurations, for which $\mathcal{F} = - *_4 \mathcal{F}$, we can rewrite it in the form

$$\Xi[(X_i, a), (\bar{X}_{\bar{k}}, \bar{b})]|_{\text{susy}} = - \int_\Sigma e^{2A - \Phi} a \wedge \bar{b} \wedge P[J] +$$
$$- \frac{i}{8} \int_\Sigma e^{2A - \Phi} \left(1 + \frac{1}{2} \mathcal{F}^2 \right) P[\imath_{X_i} \Omega \wedge \imath_{\bar{X}_{\bar{k}}} \bar{\Omega}] \, , \tag{10.28}$$

where the indexes in $\mathcal{F}^2$ are contracted with the induced metric $P[g]$.

As a further application, from the formulas (3.6) and (3.8) we can also find a general formula for the flux-induced physical mass term for the embedding moduli t^i around a supersymmetric configuration $(\Sigma_0, \mathcal{F}_0)$. From (3.6), one immediately obtains that the metric for the embedding holomorphic deformations is given by

$$G_{i\bar{k}} = \frac{1}{8} \int_\Sigma e^{2A - \Phi} \left(1 + \frac{1}{2} \mathcal{F}^2 \right) P_{\Sigma_0}[\imath_{X_i} \Omega \wedge \imath_{\bar{X}_{\bar{k}}} \bar{\Omega}] \, . \tag{10.29}$$

In the approximation in which the warp-factor, the non-trivial dilaton and $\mathcal{F}$ can be neglected, this metric reduces to the Kähler metric for the embedding moduli found in [31]. Thus, from (3.8) and reintroducing the tension as described after (4.14), we obtain the following quadratic term in the potential

$$\mathcal{V} \simeq M_{i\bar{k}}^2 (t - t_0)^i (\bar{t} - \bar{t}_0)^{\bar{k}} + \cdots , \tag{10.30}$$

where the physical mass matrix M^2 is given by

$$M_{i\bar{k}}^2 = G^{r\bar{s}}(t_0) m_{ir}(t_0) \bar{m}_{\bar{k}\bar{s}}(t_0) . \tag{10.31}$$

Thus, the superpotential generates flux-induced mass terms for the embedding moduli of the D7-branes in a general type B background. Analogous massive terms where computed in [32] by a different argument in the subcase of internal warped Calabi-Yau spaces and $\mathcal{F}_0 = 0$.

To be even more concrete, we could consider the simplified case when the divisor Σ has trivial canonical bundle and we can write $M \simeq \Sigma \times \mathbb{C}$ globally on a cylindrical neighborhood of Σ_0, in such a way that $J^{(K)} = P_\Sigma[J^{(K)}] + idt \wedge d\bar{t}$, where t is the holomorphic transverse coordinate in $\mathbb{C}$. In this case the only holomorphic embedding deformation given by the position t of Σ in $\mathbb{C}$. Let us introduce the holomorphic (2,0) form ω on Σ such that $\tilde{\Omega} = \omega \wedge dt$. Then we have that

$$G_{t\bar{t}} = \int_\Sigma e^\Phi \left(1 + \frac{1}{2} \mathcal{F}^2 \right) dVol_4 ,$$
$$m_{tt} = -\frac{ig_s}{4} \int_\Sigma e^\Phi \omega \wedge P[\imath_{\partial_t} \bar{G}_{(3)}] . \tag{10.32}$$

Thus, by posing $t = 2\pi\alpha'\phi$ and $g_{\mathrm{YM}}^2 = (2\pi)^5 (\alpha')^2 g_s / G_{t\bar{t}}$, we can write the following canonical quadratic Lagrangian for ϕ

$$\mathcal{L} = -\frac{1}{g_{\mathrm{YM}}^2} (\partial_\mu \phi \partial^\mu \bar{\phi} + M^2 \phi \bar{\phi}) . \tag{10.33}$$

where $M^2 = G_{t\bar{t}}^{-2} |m_{tt}|^2$. As a check, we can consider the simplest case where the internal manifold is a six torus $(\mathbb{T}^2)^3$, and assume that the warped factor and the $G_{(3)}$ are constant on the wrapped $\Sigma = (\mathbb{T}^2)^2$. In this case, defining $S = 2G_{12\bar{t}}$, we can write

$$g_{\mathrm{YM}}^2 = \frac{(2\pi)^5 (\alpha')^2 g_s}{Vol_4(\Sigma) - \frac{1}{2} \int_\Sigma \mathcal{F} \wedge \mathcal{F}} , \qquad M^2 = \frac{g_s^2 e^{8A_0} |S|^2}{8 + 2 \left(\frac{\int_\Sigma \mathcal{F} \wedge \mathcal{F}}{Vol_4(\Sigma)} \right)^2} . \tag{10.34}$$

The result (10.34) provides the generalization to arbitrary $\mathcal{F} \neq 0$ of the supersymmetric massive term found in [29] in the case $\mathcal{F} = 0$ by direct dimensional reduction of the D7-brane action.

Finally, if we further assume that we can make a gauge choice such that the B field is a globally defined (1,1)-form on $M \simeq \Sigma \times \mathbb{C}$, then the superpotential (10.20) can be written in the form

$$\mathcal{W} = \frac{1}{2} \int_\Sigma t\omega \wedge f , \tag{10.35}$$

where $f = \mathcal{F} - P[B]$ is the proper U(1) field strength on Σ. This superpotential coincides with the superpotential found in [20], for a class of F-theory backgrounds, and in [21], by dimensional reduction of the DBI plus CS D7-branes action on Calabi-Yau orientifolds. In these papers was also found a D-term of the form

$$D \sim \int_\Sigma P[J^{(\mathrm{K})}] \wedge \mathcal{F}, \tag{10.36}$$

where $\mathcal{F}$ was assumed to be harmonic. This same form can be found from our D-term

$$\mathcal{D} d\sigma^1 \wedge \cdots \wedge = P_\Sigma[e^{2A-\Phi} J] \wedge \mathcal{F}, \tag{10.37}$$

by simply noting that, expanding $\mathcal{F}$ in a base of harmonic forms, in (10.37) only the non-primitive component (proportional to $P_\Sigma[e^{2A-\Phi} J]$) of $\mathcal{F}$ survives and its contribution to our D-term is essentially given by (10.36).

11. Concluding remarks

In this paper we have approached the problem of giving a unified description of the dynamics of a general D-brane on a general $\mathcal{N} = 1$ background. In particular we have identified the F- and D-terms of the corresponding supersymmetric four-dimensional description. By introducing an appropriate metric on the configuration space, we have also shown how the resulting four-dimensional potential around a supersymmetric configuration can be written in the standard form dictated by $\mathcal{N} = 1$ supersymmetry. Furthermore we have seen how the corresponding F-flatness conditions can be derived from a superpotential that can be expressed in a universal way by using the integrable pure-spinor [5] of the underlying space, while the D-flatness condition can be seen as the vanishing of a moment map whose definition involves the *non*-integrable pure spinors.

It was possible to take the analysis on very general grounds thanks to the generalized calibrations introduced in [8]. They have not only simplified many technical steps but they have also provided an elegant geometrical interpretation of the resulting supersymmetric structure, due to their relation to the possible solitonic objects of the four-dimensional theory obtained through their D-brane realization. For example, in section 5 we have seen how the form of the superpotential $\mathcal{W}$ presented in (4.9) can be immediately guessed by using the generalized calibration $\omega^{(\mathrm{DW})}$ in (2.15) for domain wall D-branes and the well known relation between the superpotentials and BPS domain walls. The argument is completely analogous to the one used in [24, 25] to find effective closed string superpotentials, and also provides a non-trivial consistency check of our results.

Regarding the D-terms, we have discussed in section 6 how they can be related to cosmic strings, which constitute the other possible BPS solitonic objects allowed by the effective $\mathcal{N} = 1$ four-dimensional theory. In particular, using the generalized calibration $\omega^{(\mathrm{string})}$ written in (2.15), we have exactly reproduced from a purely D-brane setting the cosmic string tension obtained from effective four-dimensional arguments in [27]. This gives a strong explicit check not only of the correspondence proposed in [27] between supergravity cosmic strings and cosmic strings obtained by wrapping D-branes on internal cycles, but

also of the interpretation of the supergravity theory the authors of [27] started from as a good effective four-dimensional theory describing a brane-antibrane system coupled to the closed string sector (see also the discussion in [51]).

If on one side the supersymmetric solitons constructed from D-branes provide a physical interpretation of the effective $\mathcal{N} = 1$ structure presented in this theory, on the other side a proper understanding of the underlying mathematical structure seems to require some more effort. We have proposed some first results in this direction, by presenting an almost complex structure and a symplectic structure on the configuration space that are naturally associated to the superpotentials and the D-terms of the four-dimensional description. However, we have worked at the formal level and a deeper mathematical control of these structures would be desirable. First of all, the above structures are not trivially integrable. This is somehow expected, since the same happens even in the simpler case of branes on Calabi-Yau spaces [22]. However, in that case, restricting to the moduli space of supersymmetric branes the integrability of the complex and symplectic structure is recovered, moreover obtaining a resulting Kähler structure. This is compatible with the $\mathcal{N} = 1$ supersymmetry and in our more general case we then expect something similar when we really restrict to the moduli space of the supersymmetric configurations. This would require a better understanding of the moduli space of the generalized calibrated submanifolds of [8]. In any case, as [8] and this paper show, generalized complex geometry seems to be the right language to properly address these problems in a unified way.

The generality of the whole discussion automatically implies also the complete symmetry of the results if we pass from Type IIA to Type IIB (and vice-versa) and contemporary exchange the two pure spinors $\Psi^{\pm}$. This can be seen as a formal generalized mirror symmetry relating $\mathcal{N} = 1$ flux backgrounds [5], and it would be very interesting to try to give some more substantial arguments in favor of it (for discussions on generalized mirror symmetry see e.g. [26, 43, 57, 66 – 69]). For example, it would be interesting to address the problem starting from a SYZ approach [70], where D-branes play a central role and then our analysis could be helpful.

Finally, even if we have mainly focused on the purely theoretical aspects, we hope that the results could be useful also in concrete constructions of always more realistic models in string theory that have flux compactifications and D-branes as the essential ingredients (like for example in the KKLT proposal [71]). The explicit study of D7-branes on SU(3)-structure backgrounds presented in section 10 provides an example of how our analysis allows to reproduce and generalize some previous results in that direction.

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A. Deformations of D-branes on non-trivial B field

In order to better understand the possible infinitesimal deformations of the the generalized cycle $(\Sigma, \mathcal{F})$, let us briefly review the definition of twisted world-volume gauge field in the presence on a non-trivial B field [72, 73] on the internal manifold M.

A non-trivial B field can be seen as a connection of a gerbe on M [74]. Consider an open covering $\{\mathcal{U}_\alpha\}$ of M. Then a *gerbe* is defined by a Čech cocycle $\{g_{\alpha\beta\gamma}\}$ of maps $g_{\alpha\beta\gamma} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma \to \mathrm{U}(1)$ (the cocycle condition is given by the condition $g_{\beta\gamma\delta} g_{\alpha\gamma\delta}^{-1} g_{\alpha\beta\delta} g_{\alpha\beta\gamma}^{-1} = 1$ on any $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma \cap \mathcal{U}_\delta \neq \emptyset$). The inequivalent gerbes are then defined by elements of the second Čech cohomology group $\check{H}(M, C^\infty(\mathrm{U}(1))) \simeq H^3(M, \mathbb{Z})$. In string theory the B field defines a *connection* on a gerbe. Namely, we can take an open covering $\{\mathcal{U}_\alpha\}$ such that the B field is locally given by a two form B_α on any $\mathcal{U}_\alpha$. Then, on any twofold intersection $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ there are one-forms $\Lambda_{\alpha\beta}$ such that

$$B_\alpha - B_\beta = 2\pi\alpha' d\Lambda_{\alpha\beta} \qquad \text{on} \quad \mathcal{U}_\alpha \cap \mathcal{U}_\beta \,,$$
$$\Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha} = -i g_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma} \qquad \text{on} \quad \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma \,. \tag{A.1}$$

The globally defined three-form $H = dB$ is normalized in such a way that $[H/(2\pi)^2\alpha'] \in H^3(M, \mathbb{R})$ is the image of an integral class in $H^3(M, \mathbb{Z})$ and represents in real cohomology the characteristic class of the gerbe.

In presence of such a gerbe with connection, for a D-branes wrapping a submanifold Σ, we can take an open covering $\tilde{\mathcal{U}}_\alpha = \Sigma \cap \mathcal{U}_\alpha$ on Σ. Then, a "U(1) connection" on the D-brane is given by a set of one-forms A_α defined on $\tilde{\mathcal{U}}_\alpha$ and a set of transition functions $h_{\alpha\beta} : \tilde{\mathcal{U}}_\alpha \cap \tilde{\mathcal{U}}_\beta \to \mathrm{U}(1)$ such that

$$A_\alpha - A_\beta + i h_{\alpha\beta}^{-1} dh_{\alpha\beta} = P_\Sigma[\Lambda_{\alpha\beta}] \qquad \text{on} \quad \tilde{\mathcal{U}}_\alpha \cap \tilde{\mathcal{U}}_\beta \,,$$
$$h_{\alpha\beta} h_{\beta\gamma} h_{\gamma\alpha} = P_\Sigma[g_{\alpha\beta\gamma}] \qquad \text{on} \quad \tilde{\mathcal{U}}_\alpha \cap \tilde{\mathcal{U}}_\beta \cap \tilde{\mathcal{U}}_\gamma \,. \tag{A.2}$$

The world-volume globally defined field strength is given by $\mathcal{F} = 2\pi\alpha' dA_\alpha + P_\Sigma[B_\alpha]$, and obeys the modified Bianchi identity $d\mathcal{F} = P_\Sigma[H]$.

Now, consider any other U(1) connection A'_α with transition functions $h'_{\alpha\beta}$, on the same cycle Σ. Then the set of one-forms $a_\alpha/2\pi\alpha' = A'_\alpha - A_\alpha$ define a proper connection on the line bundle on Σ defined by the transitions functions $g_{\alpha\beta} = h'_{\alpha\beta} h_{\alpha\beta}^{-1}$ (such that $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$). If we consider an infinitesimal deformation of the (twisted) U(1) connection A, it is described by a globally defined 1-form a on Σ, which can be seen as a connection on the trivial line bundle on Σ. The corresponding infinitesimal deformation of the world-volume field strength is given by $\delta\mathcal{F} = da$.

Till now we have kept fixed the cycle Σ wrapped by the brane. However, we can consider also a deformation of it, generated by a vector field $X \in \Gamma(T_N)$, where $N \subset M$ is an open neighborhood of Σ. Obviously, under such a deformation, the background gerbe transition functions $g_{\alpha\beta\gamma}$ are deformed to new $g'_{\alpha\beta\gamma} \simeq g_{\alpha\beta\gamma} + \mathcal{L}_X g_{\alpha\beta\gamma}$, together with a new

gerbe connection defined by $B'_\alpha \simeq B_\alpha + \mathcal{L}_X B_\alpha$ and $\Lambda'_{\alpha\beta} \simeq \Lambda_{\alpha\beta} + \mathcal{L}_X \Lambda_{\alpha\beta}$. It is clear that also the transition functions $h_{\alpha\beta}$ which enter the definition of world-volume gauge connection in (A.2) must transform accordingly. Using (A.1) it is not difficult to see that $h'_{\alpha\beta} \simeq h_{\alpha\beta}(1 + iP_\Sigma[\imath_X \Lambda_{\alpha\beta}])$. Thus, also the gauge field A must be deformed to $A'_\alpha = A_\alpha - P_\Sigma[\imath_X B_\alpha]/2\pi\alpha'$. Note that the new gerbe defined by the transition functions $\{g'_{\alpha\beta\gamma}\}$ is related to the gerbe defined by $\{g_{\alpha\beta\gamma}\}$ through the "gauge transformation" $g'_{\alpha\beta\gamma} = g_{\alpha\beta\gamma}(f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha})$, where $f_{\alpha\beta} \simeq 1 + i\imath_X \Lambda_{\alpha\beta}$. We can then perform a further gauge transformation $B'_\alpha \to \tilde{B}_\alpha = B'_\alpha - d(\imath_X B_\alpha) \simeq B_\alpha + \imath_X H$, obtaining a new connection defined by the new $\tilde{B}_\alpha$'s for the undeformed gerbe defined by the transition functions $g_{\alpha\beta\gamma}$ and with the same transition one-forms $\Lambda_{\alpha\beta}$. Note also that the gauge transformation of the B field $\delta B_\alpha = d\imath_X B_\alpha$ turns the world-volume gauge connection back to the initial A_α with undeformed transition functions $h_{\alpha\beta}$. Then, supplemented by the gauge transformation, in this form the diffeomorfism generated by X acts only on the B field (and consequently on H) accordingly to the rule $\delta_X B = \imath_X H$, leaving all the transition functions and the world-volume gauge field untouched. The resulting infinitesimal deformation of the world-volume field strength is given by $\delta \mathcal{F} = P_\Sigma[\imath_X H]$.

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